APPENDIX TO THE PAPER "ON THE PROJECTIVE EMBEDDINGS OF GORENSTEIN TORIC DEL PEZZO SURFACES"

T. KIKUCHI AND T. NAKANO

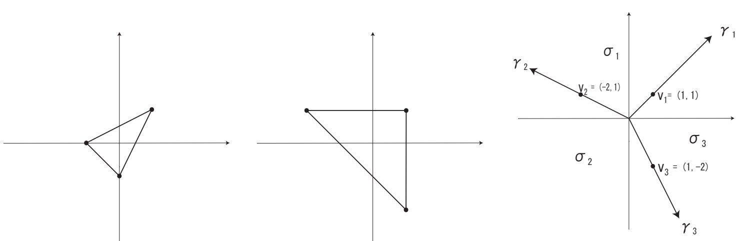
Appendix

This is an appendix to our paper "On the projective embeddings of Gorenstein toric del Pezzo surfaces". In this appendix, we summarize the computational data of the 16 complete toric Gorenstein del Pezzo surfaces $S_1, \cdots, S_{16}$ for the proof of Theorem 2.1 of the paper. For the notations and terminology, see the main text. In the following, we write CDiv(S), PDiv(S) instead of T-CDiv(S), T-PDiv(S) etc. for short (thus CDiv(S) is the group of T-stable Cartier divisors of S and PDiv(S) the subgroup of principal ones).

In the case of S_i for $i = 10, 11, 13, 14, 16$, S_i is smooth and is a blowing-up of $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$ at two points at most. In these five cases, the computation is easily done without toric geometry.

We also note that the defining equations of the minimal embedding given below are minimal in the sense that no member of them cannot be omitted. But in the non complete intersection cases, we do not know if the number of the defining equations are minimal or not.

(1) $S = S_1$ case:



R_1 $\quad$ $R_1{}^\circ$ $\quad$ Δ_1

- CDiv(S) $= \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \mathbf{Z}f_3$, where $f_1 = (1, -1, 2, 1, 0, 0)$, $f_2 = (1, 0, 1, 0, 1, 0)$, $f_3 = (-1, 2, -2, 0, 0, 1)$.

Date: August 12, 2009.
1991 *Mathematics Subject Classification.* 14J25, 14M25.

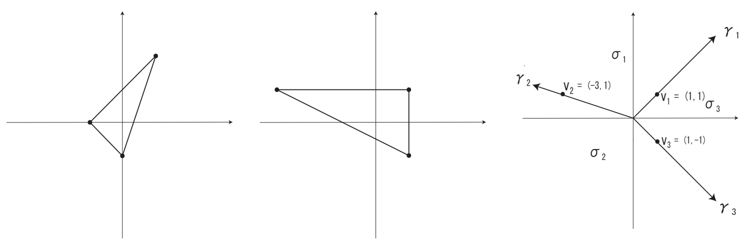
PDiv(S) $=$ $\mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, where $e_1 = (1,0,1,0,1,0) = f_2, e_2 = (0,1,0,1,0,1) = f_1 + f_3$.
Pic$(S) = $ CDiv$(S)/$PDiv$(S) \simeq \mathbf{Z}\mathcal{O}(-f_3)$.
$-K_S = \mathcal{O}((0,-1,1,1,-1,0)) = \mathcal{O}(f_1 - f_2) = \mathcal{O}(-f_3)$, index $= 1$.

- $((D_i, D_j)) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$, $[-f_3] = D_1 + 4D_2 - 2D_3$, $[-f_3]^2 = (D_1 + 4D_2 - 2D_3)^2 = 3$.

- $h^0(m\mathcal{O}(-f_3)) = \dfrac{3}{2}m^2 + \dfrac{3}{2}m + 1$, $\deg(m\mathcal{O}(-f_3)) = 3m^2$, AC$(S) = \{a > 0 \,|\, a \in \mathbf{Z}\}$.
$\min h^0 = 4$ at $m = 1 = -K_S$, $\min\deg = 3$ at $m = 1 = -K_S$.

- minimal embedding: $S_1 = \{x_0^3 - x_1x_2x_3 = 0\} \subset \mathbf{P}^3$, with 3 A_2 singularities.

(2) $S = S_2$ case:



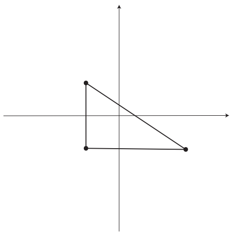
R_2 $\qquad$ $R_2{}^\circ$ $\qquad$ Δ_2

- CDiv$(S) = \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \mathbf{Z}f_3$, where $f_1 = (1,-1,2,2,0,0)$, $f_2 = (0,1,-1,-2,1,0)$, $f_3 = (0,1,0,1,0,1)$.
PDiv$(S) = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, where $e_1 = (1,0,1,0,1,0) = f_1 + f_2, e_2 = (0,1,0,1,0,1) = f_3$.
Pic$(S) = $ CDiv$(S)/$PDiv$(S) \simeq \mathbf{Z}\mathcal{O}(f_1)$.
$-K_S = \mathcal{O}((0,-1,1,2,-1,0)) = \mathcal{O}(-f_2) = \mathcal{O}(f_1)$, index $= 1$.

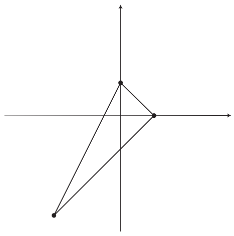
- $((D_i, D_j)) = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$, $[f_1] = 4D_2$, $[f_1]^2 = 16(D_2)^2 = 4$.
- $h^0(m\mathcal{O}(f_1)) = 2m^2 + 2m + 1$, $\deg(m\mathcal{O}(f_1)) = 4m^2$, AC$(S) = \{a > 0 | a \in \mathbf{Z}\}$.
$\min h^0 = 5$ at $m = 1 = -K_S$, $\min\deg = 4$ at $m = 1 = -K_S$.

- minimal embedding $S_2 = \{x_1x_3 - x_2{}^2 = 0, x_2x_4 - x_0{}^2 = 0\} \subset \mathbf{P}^4$, with 1 A_3-singularity and 2 A_1-singularities.

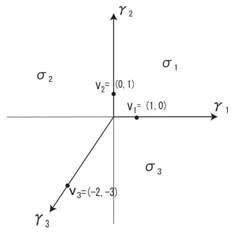
(3) $S = S_3$ case:



R_3



$R_3{}^{\circ}$



Δ_3

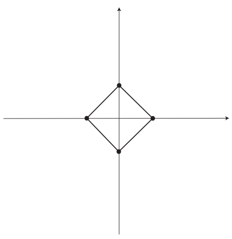
- $\mathrm{CDiv}(S_2) = \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \mathbf{Z}f_3$, where $f_1 = (0,-2,3,-2,0,0), f_2 = (1,0,1,0,1,0), f_3 = (0,-1,3,-1,0,1)$.
  $\mathrm{PDiv}(S_2) = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, where $e_1 = (1,0,1,0,1,0) = f_2, e_2 = (0,1,0,1,0,1) = -f_1 + f_3$.
  $\mathrm{Pic}(S) = \mathrm{CDiv}(S)/\mathrm{PDiv}(S) \simeq \mathbf{Z}\mathcal{O}(f_3)$.
  $-K_S = \mathcal{O}((-1,-1,2,-1,-1,1)) = \mathcal{O}(-f_2+f_3) = \mathcal{O}(f_3)$, index $= 1$.

- $((D_i, D_j)) = \begin{pmatrix} 2/3 & 1 & 1/3 \\ 1 & 3/2 & 1/2 \\ 1/3 & 1/2 & 1/6 \end{pmatrix}$, $[f_3] = D_2 + 3D_3$, $[f_3]^2 = 6$.

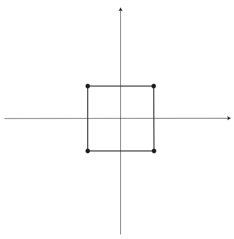
- $h^0(m\mathcal{O}(f_3)) = 3m^2 + 3m + 1$, $\deg(m\mathcal{O}(f_3)) = 6m^2$, $\mathrm{AC}(S) = \{a > 0 | a \in \mathbf{Z}\}$.
  $\min h^0 = 7$ at $m = 1 = -K_S$, $\min \deg = 6$ at $m = 1 = -K_S$.

- minimal embedding $S_3 = \{x_1x_3 - x_2{}^2 = 0, x_1x_4 - x_2x_0 = 0, x_1x_5 - x_0^2 = 0, x_2x_4 - x_3x_0 = 0, x_2x_5 - x_4x_0 = 0, x_2x_6 - x_5x_0 = 0, x_3x_5 - x_4{}^2 = 0, x_3x_6 - x_4x_5 = 0, x_4x_6 - x_5{}^2 = 0\} \subset \mathbf{P}^6$, with 1 A_1-singularity and 1 A_2-singularity.

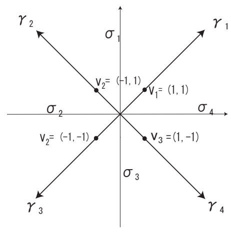
(4) $S = S_4$ case:



R_4



$R_4{}^{\circ}$



Δ_4

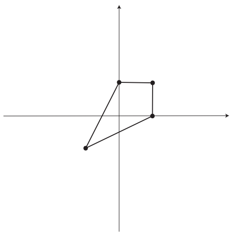
- $\mathrm{CDiv}(S) = \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \mathbf{Z}f_3 \oplus \mathbf{Z}f_4$, where $f_1 = (-1,1,-1,1,0,0,0,0), f_2 = (0,0,1,1,1,1,0,0), f_3 = (1,0,1,0,1,0,1,0), f_4 = (1,0,0,-1,-1,0,0,1)$.
  $\mathrm{PDiv}(S) = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, where $e_1 = (1,0,1,0,1,0,1,0) = f_3, e_2 = (0,1,0,1,0,10,1) = f_1 + f_2 + f_4$.
  $\mathrm{Pic}(S) = \mathrm{CDiv}(S)/\mathrm{PDiv}(S) \simeq \mathbf{Z}\mathcal{O}(f_1) \oplus \mathbf{Z}\mathcal{O}(f_4)$.
  $-K_S = \mathcal{O}((0,-1,1,0,0,1,-1,0)) = \mathcal{O}(-f_1+f_2-f_3) = -2\mathcal{O}(f_1) - \mathcal{O}(f_4)$, index $= 1$.

- $((D_i, D_j)) = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$, $[f_1] = -2D_2$, $[f_4] = -D_1+D_2-D_3+D_4$, $[f_1]^2 = 0, [f_1][f_4] = 2, [f_4]^2 = -4$.

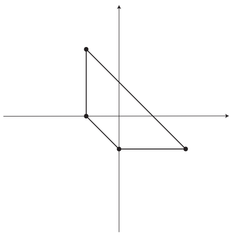
- $h^0(a\mathcal{O}(f_1) + b\mathcal{O}(f_4)) = 1 - a + 2ab - 2b^2$, $\deg(a\mathcal{O}(f_1) + b\mathcal{O}(f_4)) = 4ab - 4b^2$, $\mathrm{AC}(S) = \{(a,b) \in \mathbf{Z}^2 \mid -a+b > 0, -b > 0\}$.
  $\min h^0 = 5$ at $(a,b) = (-2,-1) = -K_S$, $\min \deg = 4$ at $(a,b) = (-2,-1) = -K_S$.

- minimal embedding $S_4 = \{x_1x_3 - {x_0}^2 = 0, x_2x_4 - {x_0}^2 = 0\} \subset \mathbf{P}^4$, with 4 A_1singularities.

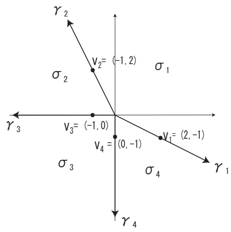
(5) $S = S_5$ case:



R_5



$R_5{}^\circ$



Δ_5

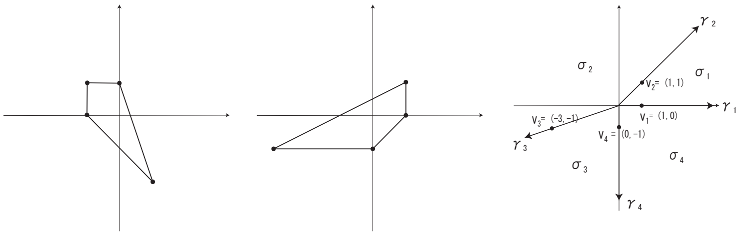
- $\mathrm{CDiv}(S) = \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \mathbf{Z}f_3 \oplus \mathbf{Z}f_4$, where $f_1 = (1,2,-3,0,-3,0,0,0), f_2 = (0,0,2,1,2,0,0,0), f_3 = (0,-2,4,0,4,0,1,0), f_4 = (0,1,-2,0,-2,1,0,1)$.
  $\mathrm{PDiv}(S) = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, where $e_1 = (1,0,1,0,1,0,1,0) = f_1 + f_3$,
  $e_2 = (0,1,0,1,0,1,0,1) = f_2 + f_4$.
  $\mathrm{Pic}(S) = \mathrm{CDiv}(S)/\mathrm{PDiv}(S) \simeq \mathbf{Z}\mathcal{O}(f_3) \oplus \mathbf{Z}\mathcal{O}(f_4)$.
  $-K_S = \mathcal{O}((-1,-1,1,0,1,1,0,1)) = \mathcal{O}(-f_1+f_4) = \mathcal{O}(f_3) + \mathcal{O}(f_4)$, index $= 1$.

- $((D_i, D_j)) = \begin{pmatrix} 1/6 & 1/2 & 0 & 1/3 \\ 1/2 & -1/2 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \\ 1/3 & 0 & 1/2 & 1/6 \end{pmatrix}$
  $[f_3] = 4D_1 + 4D_2 - 2D_4$, $[f_4] = -2D_1 - 2D_2 + D_3 + D_4$, $[f_3]^2 = 6, [f_3][f_4] = 0, [f_4]^2 = -2$.

- $h^0(a\mathcal{O}(f_3) + b\mathcal{O}(f_4)) = 1 + 3a - b + 3a^2 - b^2$, $\deg(a\mathcal{O}(f_3) + b\mathcal{O}(f_4)) = 6a^2 - 2b^2$, $\mathrm{AC}(S) = \{(a,b) \in \mathbf{Z}^2 \mid 2a - b > 0, b > 0, 3a - 2b > 0, a > 0\}$.
  $\min h^0 = 5$ at $(a,b) = (1,1) = -K_S$, $\min \deg = 4$ at $(a,b) = (1,1) = -K_S$.

- minimal embedding $S_5 = \{x_1x_2 - x_3x_0 = 0, x_3x_4 - {x_0}^2 = 0\} \subset \mathbf{P}^4$, with 2 A_1-singularities and a A_2-singularity.

(6) $S = S_6$ case



R_6 $\quad$ $R_6{}^\circ$ $\quad$ Δ_6

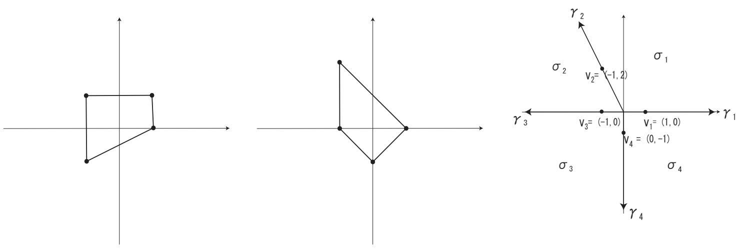
- $\mathrm{CDiv}(S) = \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \mathbf{Z}f_3 \oplus \mathbf{Z}f_4$, where $f_1 = (0, 3, 0, 3, 1, 0, 0, 0)$, $f_2 = (0, 1, 0, 1, 0, 1, 0, 1), f_3 = (0, -2, 1, -3, 0, 0, 0, 0), f_4 = (1, -1, 0, 0, 0, 0, 1, 0)$.
$\mathrm{PDiv}(S) = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, where $e_1 = (1, 0, 1, 0, 1, 0, 1, 0) = f_1 + f_3 + f_4, e_2 = (0, 1, 0, 1, 0, 1, 0, 1) = f_2$.
$\mathrm{Pic}(S) = \mathrm{CDiv}(S)/\mathrm{PDiv}(S) = \mathbf{Z}\mathcal{O}(f_3) \oplus \mathbf{Z}\mathcal{O}(f_4)$.
$-K_S = \mathcal{O}((-1, 0, 1, -2, 0, 1, -1, 1)) = \mathcal{O}(f_2 + f_3 - f_4) = \mathcal{O}(f_3) - \mathcal{O}(f_4)$, index $= 1$.

- $((D_i, D_j)) = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/6 & 1/3 \\ 1 & 0 & 1/3 & -1/3 \end{pmatrix}$, $[f_3] = 2D_2$, $[f_4] = -D_1$, $[f_3]^2 = 2, [f_3][f_4] = -2, [f_4]^2 = -1$.

- $h^0(a\mathcal{O}(f_3) + b\mathcal{O}(f_4)) = 1 + 2a - b/2 - 2ab + a^2 - (1/2)b^2$, $\deg(a\mathcal{O}(f_3) + b\mathcal{O}(f_4)) = 2a^2 - 4ab - b^2$.
$\mathrm{AC}(S) = \{(a, b) \in \mathbf{Z}^2 \mid 2a + b > 0, a - b > 0, a > 0, -b > 0\}$.
$\min h^0 = 6$ at $(a, b) = (1, -1) = -K_S$, $\min \deg = 5$ at $(a, b) = (1, -1) = -K_S$.

- minimal embedding $S_6 = \{x_1x_3 - x_2x_0 = 0, x_1x_4 - {x_0}^2 = 0, x_2x_4 - x_3x_0 = 0, x_2x_5 - x_4x_0 = 0, x_3x_5 - {x_4}^2 = 0\} \subset \mathbf{P}^5$, with 1 A_1-singularity and 1 A_2-singularity.

(7) $S = S_7$ case:



R_7 $\quad$ $R_7{}^\circ$ $\quad$ Δ_7

- $\mathrm{CDiv}(S) = \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \mathbf{Z}f_3 \oplus \mathbf{Z}f_4$, where $f_1 = (-2, 0, 0, 1, 0, 0, -2, 0), f_2 = (2, 1, 0, 0, 0, 0, 2, 0), f_3 = (1, 0, 1, 0, 1, 0, 1, 0), f_4 = (0, 0, 0, 0, 0, 1, 0, 1)$.
$\mathrm{PDiv}(S) = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, where $e_1 = (1, 0, 1, 0, 1, 0, 1, 0) = f_3, e_2 = (0, 1, 0, 1, 0, 1, 0, 1) = f_1 + f_2 + f_4$.
$\mathrm{Pic}(S) = \mathrm{CDiv}(S)/\mathrm{PDiv}(S) = \mathbf{Z}\mathcal{O}(f_1) \oplus \mathbf{Z}\mathcal{O}(f_4)$.
$-K_S = \mathcal{O}((-1, -1, 1, 0, 1, 1, -1, 1)) = \mathcal{O}(-f_2 + f_3 + f_4) = \mathcal{O}(f_1) + 2\mathcal{O}(f_4)$, index

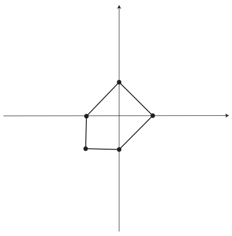
$= 1.$

- $((D_i, D_j)) = \begin{pmatrix} 1/2 & 1/2 & 0 & 1 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & -1/2 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $[f_1] = 2D_1 - 2D_2$, $[f_4] = D_4$, $[f_1]^2 = -2, [f_1][f_4] = 2, [f_4]^2 = 0$.

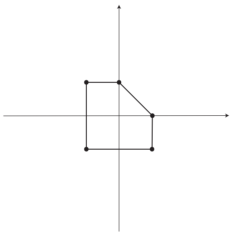
- $h^0(a\mathcal{O}(f_1) + b\mathcal{O}(f_4)) = 1 + a + b - a^2 + 2ab$, $\deg(a\mathcal{O}(f_1) + b\mathcal{O}(f_4)) = -2a^2 + 4ab$.
  $\mathrm{AC}(S) = \{(a, b) \in \mathbf{Z}^2 \mid b > 0, a > 0, -a + b > 0\}$.
  $\min h^0 = 7$ at $(a, b) = (1, 2) = -K_S$, $\min \deg = 6$ at $(a, b) = (1, 2) = -K_S$.

- minimal embedding $S_7 = \{x_1x_2 - x_3x_0 = 0, x_1x_4 - x_2x_0 = 0, x_1x_5 - {x_0}^2 = 0, {x_2}^2 - x_3x_4 = 0, x_2x_5 - x_4x_0 = 0, x_2x_6 - x_5x_0 = 0, x_3x_5 - x_2x_0 = 0, x_3x_6 - {x_0}^2 = 0, x_4x_6 - {x_5}^2 = 0\} \subset \mathbf{P}^6$, with 2 A_1-singularities.

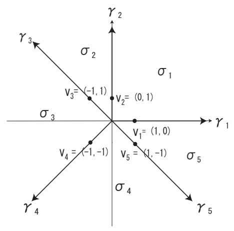
(8) $S = S_8$ case:



R_8



$R_8{}^\circ$



Δ_8

- $\mathrm{CDiv}(S) = \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \mathbf{Z}f_3 \oplus \mathbf{Z}f_4 \oplus \mathbf{Z}f_5$, where $f_1 = (2, 0, 0, 0, 1, 1, 2, 0, 2, 0)$, $f_2 = (-1, 1, 0, 1, -1, 0, -1, 0, -1, 0)$, $f_3 = (-2, 0, 0, 0, 0, 0, -1, 1, -2, 0)$,
  $f_4 = (1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$, $f_5 = (1, 0, 0, 0, 0, 0, 0, 0, 1, 1)$.
  $\mathrm{PDiv}(S) = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, where $e_1 = (1, 0, 1, 0, 1, 0, 1, 0, 1, 0) = f_4$,
  $e_2 = (0, 1, 0, 1, 0, 1, 0, 1, 0, 1) = f_1 + f_2 + f_3 + f_5$.
  $\mathrm{Pic}(S) = \mathrm{CDiv}(S)/\mathrm{PDiv}(S_1) = \mathbf{Z}\mathcal{O}(f_1) \oplus \mathbf{Z}\mathcal{O}(f_3) \oplus \mathbf{Z}\mathcal{O}(f_5)$.
  $-K_S = \mathcal{O}((-1, -1, 0, -1, 1, 0, 0, 1, -1, 0)) = \mathcal{O}(-f_2 + f_3) = \mathcal{O}(f_1) + 2\mathcal{O}(f_3) + \mathcal{O}(f_5)$, index $= 1$.

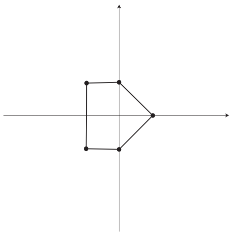
- $((D_i, D_j)) = \begin{pmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1 & 0 & 0 & 1/2 & -1/2 \end{pmatrix}$, $[f_1] = -2D_1 + 2D_4 - 2D_5$, $[f_3] = 2D_1 + 2D_5$, $[f_5] = -D_1$.
  For orthogonalization $F_1 := \mathcal{O}(f_3) + 2\mathcal{O}(f_5), F_2 := \mathcal{O}(f_1), F_3 := -\mathcal{O}(f_3) - \mathcal{O}(f_5)$.
  Then ${F_1}^2 = {F_2}^2 = -2, {F_3}^2 = 1$, $(F_i, F_j) = 0\,(i \neq j)$. $-K_S = -F_1 + F_2 - 3F_3$.

- $h^0(aF_1 + bF_2 + cF_3) = 1 + a - b - (3/2)c - a^2 - b^2 + (1/2)c^2$, $\deg(aF_1 + bF_2 + cF_3) = -2a^2 - 2b^2 + c^2$.
  $\mathrm{AC}(S) = \{(a, b, c) \in \mathbf{Z}^3 \mid -a > 0, b > 0, 2a - c > 0, -2b - c > 0\}$.
  $\min h^0 = 6$ at $(a, b, c) = (-1, 1, -3) = -K_S$, $\min \deg = 5$ at $(a, b, c) = (-1, 1, -3) =$

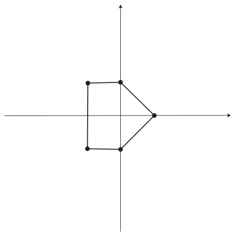
$-K_S$.

- minimal embedding $S_8 = \{x_1x_3 - {x_0}^2 = 0, x_1x_5 - x_4x_0 = 0, x_2x_4 - {x_0}^2 = 0, x_2x_5 - x_3x_0 = 0, x_3x_4 - x_5x_0 = 0\} \subset \mathbf{P}^5$, with 2 A_1-singularities.

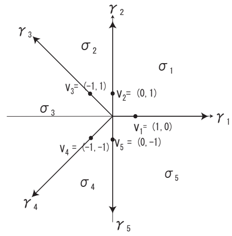
(9) $S = S_9$ case:



R_9



${R_9}^\circ$



Δ_9

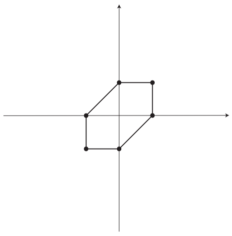
- $\mathrm{CDiv}(S) = \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \mathbf{Z}f_3 \oplus \mathbf{Z}f_4 \oplus \mathbf{Z}f_5$, where $f_1 = (0,2,0,2,-1,1,0,0,0,0)$, $f_2 = (0,-1,0,-1,1,0,1,0,0,0)$, $f_3 = (0,-1,0,-1,1,0,0,1,0,1)$,
  $f_4 = (0,1,1,1,0,0,0,0,0,0)$, $f_5 = (1,0,0,0,0,0,0,0,1,0)$.
  $\mathrm{PDiv}(S) = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, where $e_1 = (1,0,1,0,1,0,1,0,1,0) = f_2 + f_4 + f_5, e_2 = (0,1,0,1,0,1,0,1,0,1) = f_1 + f_3$.
  $\mathrm{Pic}(S) = \mathrm{CDiv}(S)/\mathrm{PDiv}(S) = \mathbf{Z}\mathcal{O}(f_3) \oplus \mathbf{Z}\mathcal{O}(f_4) \oplus \mathbf{Z}\mathcal{O}(f_5)$.
  $-K_S = \mathcal{O}((-1,-1,0,-1,1,0,0,1,-1,1)) = \mathcal{O}(f_3) - \mathcal{O}(f_5)$, index $= 1$.

- $((D_i, D_j)) = \begin{pmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$, $[f_3] = D_1 + D_2 + D_3 + D_4$, $[f_4] = -D_1$, $[f_5] = -D_5$.
  For orthogonalization $F_1 := \mathcal{O}(f_3), F_2 := \mathcal{O}(f_4), F_3 := -\mathcal{O}(f_3) - \mathcal{O}(f_4) - \mathcal{O}(f_5)$.
  Then ${F_1}^2 = 2, {F_2}^2 = {F_3}^2 = -1$, $(F_i, F_j) = 0\,(i \neq j)$. $-K_S = 2F_1 + F_2 + F_3$.

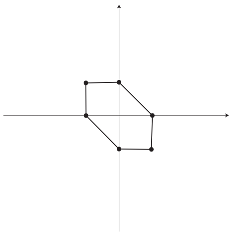
- $h^0(aF_1 + bF_2 + cF_3) = 1 + 2a - (1/2)b - (1/2)c + a^2 - (1/2)b^2 - (1/2)c^2$, $\deg(aF_1 + bf_2 + cF_3) = 2a^2 - b^2 - c^2$.
  $\mathrm{AC}(S) = \{(a,b,c) \in \mathbf{Z}^3 \mid b > 0, a - b > 0, a - c > 0, c > 0, 2a - b - c > 0\}$.
  $\min h^0 = 7$ at $(a,b,c) = (2,1,1) = -K_S$, $\min \deg = 6$ at $(a,b,c) = (2,1,1) = -K_S$.

- minimal embedding $S_9 = \{x_1x_3 - x_2x_0 = 0, x_1x_4 - {x_0}^2 = 0, x_1x_5 - x_6x_0 = 0, x_2x_4 - x_3x_0 = 0, x_2x_5 - x_4x_0 = 0, x_2x_6 - {x_0}^2 = 0, x_3x_5 - {x_4}^2 = 0, x_3x_6 - x_4x_0 = 0, x_4x_6 - x_5x_0 = 0\} \subset \mathbf{P}^6$, with 1 A_1-singularity.

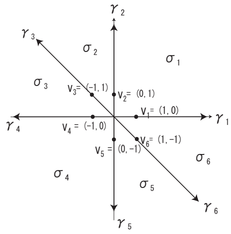
(10) $S = S_{10}$ case:



R_{10}



$R_{10}{}^{\circ}$



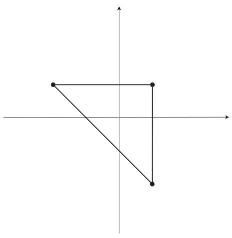
Δ_{10}

- $S = S_{10} \cong Q_{P_1P_2}(\mathbf{P}^1 \times \mathbf{P}^1)$, the blowing-up of $\mathbf{P}^1 \times \mathbf{P}^1$ at two points P_1, P_2 not lying on a ruling line. Let $\pi : S \to \mathbf{P}^1 \times \mathbf{P}^1$ be the projection and E_i the line bundle corresponding to the exceptional curve $\pi^{-1}(P_i)$ $(i = 1, 2)$. $\mathrm{Pic}(\mathbf{P}^1 \times \mathbf{P}^1) = \mathbf{Z}l_1 \oplus \mathbf{Z}l_2$, where $l_i := p_i^*(\mathcal{O}_{\mathbf{P}^1}(1))$, $p_i : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ the projection to the i-th factor $(i = 1, 2)$. Set $L_i := \pi^*(l_i) \in \mathrm{Pic}(S)$. Then $\mathrm{Pic}(S) = \mathbf{Z}L_1 \oplus \mathbf{Z}L_2 \oplus \mathbf{Z}E_1 \oplus \mathbf{Z}E_2$, where ${L_1}^2 = {L_2}^2 = 0, {E_1}^2 = {E_2}^2 = -1, (L_1, L_2) = 1, (E_1, E_2) = 0, (L_i, E_j) = 0$ $(i, j = 1, 2)$.
  $K_S = -2L_1 - 2L_2 + E_1 + E_2$, index $= 1$.

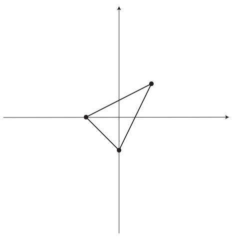
- $h^0(aL_1 + bL_2 + cE_1 + dE_2) = 1 + a + b + c/2 + d/2 + ab - c^2/2 - d^2/2$, $\deg(aL_1 + bL_2 + cE_1 + dE_2) = 2ab - c^2 - d^2$.
  $\mathrm{AC}(S) = \{(a, b, c, d) \in \mathbf{Z}^5 \mid -c > 0, -d > 0, b+c > 0, a+c > 0, b+d > 0, a+d > 0\}$.
  $\min h^0 = 7$ at $(a, b, c, d) = (2, 2, -1, -1) = -K_S$, $\min\deg = 6$ at $(a, b, c, d) = (2, 2, -1, -1) = -K_S$.

- minimal embedding $S_{10} = \{x_1x_2 - x_3x_0 = 0, x_1x_4 - {x_0}^2 = 0, x_1x_6 - x_5x_0 = 0, x_2x_5 - {x_0}^2 = 0, x_2x_6 - x_4x_0 = 0, x_3x_4 - x_2x_0 = 0, x_3x_5 - x_1x_0 = 0, x_3x_6 - {x_0}^2 = 0, x_4x_5 - x_6x_0 = 0 \subset \mathbf{P}^6$. S_{10} is smooth.

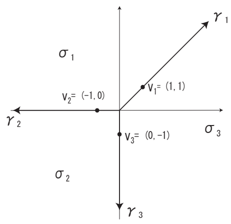
(11) $S = S_{11}$ case:



R_{11}



$R_{11}{}^{\circ}$



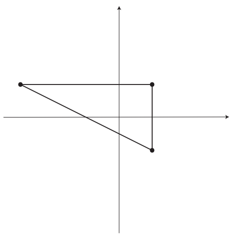
Δ_{11}

- $S = S_{11} \cong \mathbf{P}^2$.
  $\mathrm{Pic}(S) = \mathbf{Z}L$, where $L = \mathcal{O}_{\mathbf{P}^2}(1), L^2 = 1$.
  $K_S = -3L$, ${K_S}' = -L$, index $= 3$.

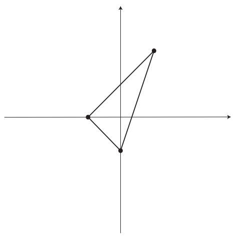
- $h^0(aL) = 1 + (3/2)a + a^2/2$, $\deg(aL) = a^2$.
  $\mathrm{AC}(S) = \{a \in \mathbf{Z} \mid a > 0\}$.
  $\min h^0 = 3$ at $a = 1 = -{K_S}'$, $\min\deg = 1$ at $a = 1 = -{K_S}'$.

- minimal embedding $S_{11} = \mathbf{P}^2$. S_{11} is smooth.

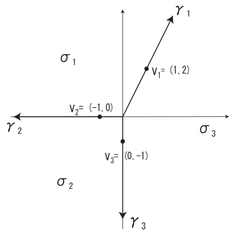
(12) $S = S_{12}$ case:



R_{12}



$R_{12}{}^{\circ}$



Δ_{12}

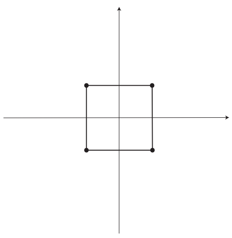
- $\mathrm{CDiv}(S) = \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \mathbf{Z}f_3$, where $f_1 = (-2, 1, -2, 0, 0, 0,), f_2 = (1, 0, 1, 0, 1, 0)$, $f_3 = (2, 0, 2, 1, 0, 1)$.
  $\mathrm{PDiv}(S) = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, where $e_1 = (1, 0, 1, 0, 1, 0) = f_2, e_2 = (0, 1, 0, 1, 0, 1)$
  $= f_1 + f_3$.
  $\mathrm{Pic}(S) = \mathrm{CDiv}(S)/\mathrm{PDiv}(S) = \mathbf{Z}\mathcal{O}(f_3)$.
  $-K_S = \mathcal{O}((1, -1, 1, 1, -3, 1)) = \mathcal{O}(-f_1 - 3f_2 + f_3) = 2\mathcal{O}(f_3)$, index $= 2$.

- $((D_i, D_j)) = \begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $[f_3] = -2D_1 + 2D_2 + D_3$, $[f_3]^2 = 2$.

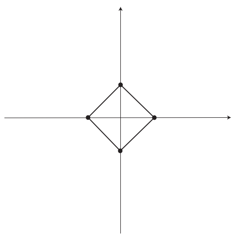
- $h^0(m\mathcal{O}(f_3)) = m^2 + 2m + 1$, $\deg(m\mathcal{O}(f_3)) = 2m^2$.
  $\mathrm{AC}(S) = \{a > 0 \,|\, a \in \mathbf{Z}\}$.
  $\min h^0 = 4$ at $m = 1 = -K_S{}'$, $\min \deg = 2$ at $m = 1 = -K_S{}'$.

- minimal embedding $S_{12} = \{x_0x_1 - x_2{}^2 = 0\} \subset \mathbf{P}^3$, with 1 A_1-singularity.

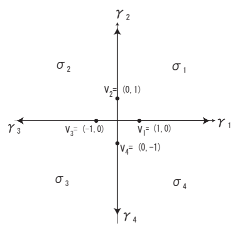
(13) $S = S_{13}$ case:



R_{13}



$R_{13}{}^{\circ}$



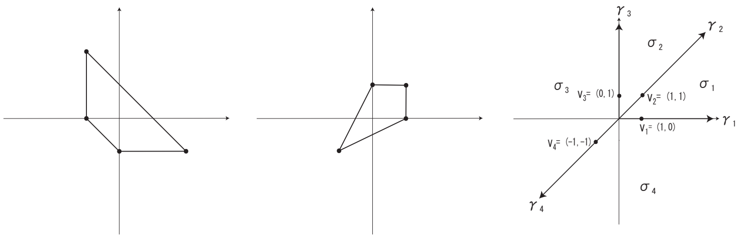
Δ_{13}

- $S = S_{13} \cong \mathbf{P}^1 \times \mathbf{P}^1$.
  $\mathrm{Pic}(S) = \mathbf{Z}L_1 \oplus \mathbf{Z}L_2$, where $L_i := p_i^*(\mathcal{O}_{\mathbf{P}^1}(1))$, $p_i : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ the projection to the i-th factor $(i = 1, 2)$.
  $K_S = -2L_1 - 2L_2$, $K_S{}' = -L_1 - L_2$, index $= 2$.

- $h^0(aL_1 + bL_2) = 1 + a + b + ab$, $\deg(aL_1 + bL_2) = 2ab$.
  $\mathrm{AC}(S) = \{(a, b) \in \mathbf{Z}^2 \,|\, a > 0, b > 0\}$.
  $\min h^0 = 4$ at $(a, b) = (1, 1) = -K_S{}'$, $\min \deg = 2$ at $(a, b) = (1, 1) = -K_S{}'$.

- minimal embedding: $S_{13} = \{x_0x_1 - x_2x_3 = 0\} \subset \mathbf{P}^3$. S_{13} is smooth.

(14) $S = S_{14}$ case:



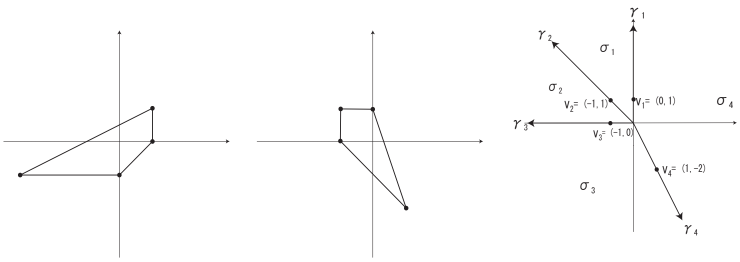
R_{14} $\quad$ ${R_{14}}^{\circ}$ $\quad$ Δ_{14}

- $S = S_{14} \cong Q_P(\mathbf{P}^2)$, the blowing-up of $\mathbf{P}^2$ at a point P.
  $\mathrm{Pic}(S) = \mathbf{Z}L \oplus \mathbf{Z}E$, where $\pi : S \to \mathbf{P}^2$ is the blowing-up at P, E is the line bundle associated to the exceptional divisor, and $L = \pi^*(\mathcal{O}_{\mathbf{P}^2}(1))$. $L^2 = 1, E^2 = -1, LE = 0$.
  $K_S = -3L + E$, index $= 1$.

- $h^0(aL + bE) = 1 + (3/2)a + b/2 + a^2/2 - b^2/2$, $\deg(aL + bE) = a^2 - b^2$.
  $\mathrm{AC}(S) = \{a, b \in \mathbf{Z}^2 \mid a + b > 0, -b > 0\}$.
  $\min h^0 = 5$ at $(a, b) = (2, -1) \neq -K_S$, $\min \deg = 3$ at $(a, b) = (2, -1) \neq -K_S$.

- minimal embedding: $S_{14} = \{x_0x_1 - x_2x_3 = 0, {x_1}^2 - x_3x_4 = 0, x_0x_4 - x_1x_2 = 0\} \subset \mathbf{P}^4$. S_{14} is smooth.

(15) $S = S_{15}$ case:



R_{15} $\quad$ ${R_{15}}^{\circ}$ $\quad$ Δ_{15}

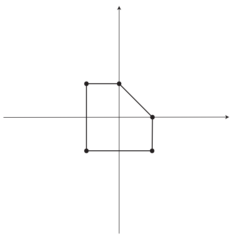
- $\mathrm{CDiv}(S) = \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \mathbf{Z}f_3 \oplus \mathbf{Z}f_4$, where $f_1 = (0, 0, 1, 1, 1, 0, 1, 0)$,
  $f_2 = (0, 0, 0, 0, 0, 1, -2, 0), f_3 = (1, 0, 0, -1, 0, 0, 0, 0), f_4 = (0, 1, 0, 1, 0, 0, 2, 1)$.
  $\mathrm{PDiv}(S) = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, where $e_1 = (1, 0, 1, 0, 1, 0, 1, 0) = f_1 + f_3$,
  $e_2 = (0, 1, 0, 1, 0, 1, 0, 1) = f_2 + f_4$.
  $\mathrm{Pic}\, S = \mathrm{CDiv}(S)/\mathrm{PDiv}(S) = \mathbf{Z}\mathcal{O}(f_3) \oplus \mathbf{Z}\mathcal{O}(f_4)$.
  $-K_S = \mathcal{O}((0, -1, 1, 0, 1, 1, -3, -1)) = \mathcal{O}(f_1 + f_2 - f_4) = -\mathcal{O}(f_3) - 2\mathcal{O}(f_4)$, index $= 1$.

- $((D_i, D_j)) = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & -1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$, $[f_3] = D_1$, $[f_4] = -D_1 - D_4$, $[f_3]^2 = -1, [f_3][f_4] = 0, [f_4]^2 = 2$.

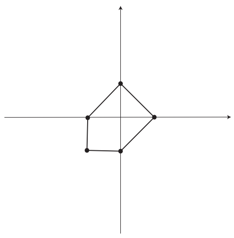
- $h^0(a\mathcal{O}(f_3)+b\mathcal{O}(f_4)) = 1+a/2-2b-a^2/2+b^2$, $\deg(a\mathcal{O}(f_3)+b\mathcal{O}(f_4)) = -a^2+2b^2$.
  $\mathrm{AC}(S) = \{(a,b) \in \mathbf{Z}^2 \mid -a > 0, a-b > 0, -b > 0, a-2b > 0\}$.
  $\min h^0 = 8$ at $(a,b) = (-1,-2) = -K_S$, $\min \deg = 7$ at $(a,b) = (-1,-2) = -K_S$.

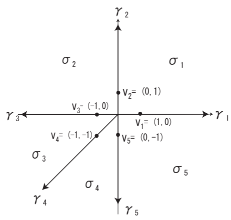
- minimal embedding $S_{15} = \{x_1x_3 - x_0{}^2 = 0, x_1x_4 - x_5x_0 = 0, x_1x_6 - x_4x_0 = 0, x_1x_7 - x_6x_0 = 0, x_2x_4 - x_0{}^2 = 0, x_2x_5 - x_1x_0 = 0, x_2x_6 - x_3x_0 = 0, x_2x_7 - x_3{}^2 = 0, x_3x_4 - x_6x_0 = 0, x_3x_5 - x_4x_0 = 0, x_3x_6 - x_7x_0 = 0, x_4{}^2 - x_5x_6 = 0, x_4x_6 - x_5x_7 = 0, x_4x_7 - x_6{}^2 = 0\} \subset \mathbf{P}^7$, with 1 A_1-singularity.

(16) $S = S_{16}$ case:



R_{16} $\quad$ $R_{16}{}^{\circ}$ $\quad$ Δ_{16}





- $S = S_{16} \cong Q_{P_1P_2}(\mathbf{P}^2)$, the blowing-up of $\mathbf{P}^2$ at two points P_1, P_2.
  $\mathrm{Pic}(S) = \mathbf{Z}L \oplus \mathbf{Z}E_1 \oplus \mathbf{Z}E_2$, where $\pi : S \to \mathbf{P}^2$ is the blowing-up, E_i is the line bundle associated to the exceptional divisor $(i = 1,2)$, and $L = \pi^*(\mathcal{O}_{\mathbf{P}^2}(1))$.
  $L^2 = 1, E_i{}^2 = -1, (E_1, E_2) = 0, (L, E_i) = 0$ $(i = 1,2)$.
  $K_S = -3L + E_1 + E_2$, index $= 1$.

- $h^0(aL+bE_1+cE_2) = 1+(1/2)(3a+b+c)+(1/2)(a^2-b^2-c^2)$, $\deg(aL+bE_1+cE_2) = a^2 - b^2 - c^2$.
  $\mathrm{AC}(S) = \{(a,b,c) \in \mathbf{Z}^3 \mid -b > 0, -c > 0, a+b+c > 0\}$.
  $\min h^0 = 8$ at $(a,b,c) = (3,-1,-1) = -K_S$, $\min \deg = 7$ at $(a,b,c) = (3,-1,-1) = -K_S$.

- minimal embedding: $S_{16} = \{x_1x_3 - x_2x_0 = 0, x_1x_4 - x_0{}^2 = 0, x_1x_5 - x_6x_0 = 0, x_1x_6 - x_7x_0 = 0, x_2x_4 - x_3x_0 = 0, x_2x_5 - x_4x_0 = 0, x_2x_6 - x_0{}^2 = 0, x_2x_7 - x_1x_0 = 0, x_3x_5 - x_4{}^2 = 0, x_3x_6 - x_4x_0 = 0, x_3x_7 - x_0{}^2 = 0, x_4x_6 - x_5x_0 = 0, x_4x_7 - x_6x_0 = 0, x_5x_7 - x_6{}^2 = 0\} \subset \mathbf{P}^7$. S_{16} is smooth.

Takashi Kikuchi and Tetsuo Nakano, Department of Mathematical Sciences, Tokyo Denki University, Hiki-gun, Hatoyama-machi, Saitame-ken, 350-0394, Japan
E-mail address: tnakano@mail.dendai.ac.jp