On the moduli space of polygons with area center

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Abstract

A point p is said to be an area center of a polygon if all of the triangles composed of p and its edges have one and the same area. We construct a moduli space AC_n of such n-gons and study its geometry and arithmetic. For every $n \geq 5$, the moduli space is proved to be a rational complete intersection subvariety in $\mathbb{A}^n$. With the help of some subvarieties of low degree in AC_n, we also find a unified method of construction of good-looking polygons with area center.
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0 Introduction

Let G be a barycenter of a triangle. Then the areas of three triangles made up of G and its edges are one and the same, as is shown by an elementary argument. In view of this fact, G is entitled to be called the *area center* of the triangle. A general n-gon, however, does not always have an area center (see Proposition 1.3). Hence it will be natural to consider when an n-gon has an area center. This seemingly innocent problem is, unexpectedly, found to have a intimate connection with the theory of Chebyshev varieties V_n developed in [2], [3]. In the present paper, we investigate the geometry and arithmetic of the moduli space, called AC_n, of n-gons with the origin as an area center. Among other things we show that AC_n is a rational complete intersection variety of codimension three in $\mathbb{A}^n$. This fact is proved by constructing a Groebner basis for its defining ideal, which in turn provides us with a parametrization of simple form for any n. Every parameter, however, does not always correspond to an n-gon with an area center of good shape. Here again we find that the family of linear subvarieties of a Chebyshev variety, constructed in [3], plays a crucial role to specify *good-looking* n-gons. Several examples of such n-gons are illustrated in the final section. Actually we need a slight generalization of the construction, and we introduce the set of strings of parentheses as well as those with *bra-ket* introduced by Dirac. The latter has no direct connection with quantum theory in this paper, but it will give us a unified viewpoint to study the set of subvarieties of low degree of AC_n.

The plan of this paper is as follows. Section one gives a precise definition of n-gons with an area center, and shows how our problem is related to the theory of Chebyshev varieties. Several useful identities are recalled and generalized, and help us to derive the defining equation of the moduli space AC_n of those n-gons. In Section two we construct a Groebner basis of the defining ideal of AC_n and show that it is a rational complete intersection variety. Furthermore we show that AC_n is nonsingular when n is not divisible by four, and in the latter case it has the origin in $\mathbb{A}^n$ as the unique singular point. The fact that the singular point of AC_4 corresponds to the square is one of amusing consequences of our results, and suggests that the geometry of the variety AC_n reflects the shapes of corresponding n-gons . From Section three on we introduce sets of strings of round brackets including an angle bracket, a bra-ket, a triple bra-ket, and a quadruple bra-ket. In Section three we recall and refine our construction of linear subvarieties of the Chebyshev varieties in [3]. We introduce the notion of *content* and of *associate polynomials* of a string, which will play a central role throughout the paper. Section four deals with the strings with angle brackets, and Section five deals with those with a bra-ket. The former gives subvarieties of V_{2n-1}, and the latter those of V_{2n}. Section six together with Section seven is devoted to the construction of subvarieties of AC_n with $n \equiv 0 \pmod 4$. Since AC_n is defined to be the intersection of three Chebyshev varieties, we need to make two other strings from a given string. The first is made by *associative transformation* (Section six) and the second by *bra-ketting transformation* (Section seven). Section eight together with Section nine is devoted to the construction of subvarieties of AC_n with $n \equiv 1 \pmod 2$. In these sections we introduce strings with a triple bra-ket and the associative transformation of the second kind. Furthermore Secion ten together with Section eleven deals with the construction of subvarieties of AC_n with $n \equiv 2 \pmod 4$. Here we introduce strings with a quadraple bra-ket and the associative transformation of the third kind. In Section twelve we illustrate our construction of good-looking n-gons with an area center for small n, and conclude the paper by finding that the most symmetric polygons are associated to the set of invariant elements under the action of the symmetric group on AC_n.

1 Polygon with area center

Let a polygon P in $\mathbb{R}^2$ have n vertices p_i for $i = 0, \cdots, n-1$, where p_k is understood to be equal to $p_{k \bmod n}$ when k is smaller than 0 or greater than $n-1$. We introduce the notion of *area-center* of P as follows:

Definition 1.1. *A point $c \in \mathbb{R}^2$ is said to be an area center of P if all the areas of the n triangles with vertices c, p_i, p_{i+1} for $i = 0, \cdots, n-1$ are one and the* *same nonzero real number.*

Remark 1.1. As is the case in this definition and throughout the paper, the area means the *signed* area.

When P is a triangle, the barycenter q of P coincides with its area-center, since the length of perpendicular from q to each side $p_i p_{i+1}$ $(0 \leq i \leq 2)$ is one third of that of perpendicular from p_{i-1}. When $n \geq 4$, however, there need not exist an area-center of a general n-gon, as is seen later. The main purpose of this paper is to investigate when an n-gon has an area-center. In order to begin our study, we express the condition that the origin is an area center of P in terms of the determinant $[p_i, p_{i+1}]$ $(0 \leq i \leq n-1)$ of 2-by-2 matrix (p_i, p_{i+1}) composed of p_i and p_{i+1}:

Proposition 1.1. *The origin of* $\mathbb{R}^2$ *is an area-center of* P *if and only if the equality*

$$[p_{i-1}, p_i] = [p_i, p_{i+1}] \tag{1.1}$$

holds for any i *with* $1 \leq i \leq n$.

Proof. This follows directly from the definition of the area-center, since $[p_{i-1}, p_i]$ is twice the area of the triangle composed of p_{i-1}, the origin, and p_i. □

Since we can express the difference $[p_{i-1}, p_i] - [p_i, p_{i+1}]$ as $[p_{i-1} + p_{i+1}, p_i]$, the condition (1.1) says that $p_{i-1} + p_{i+1}$ and p_i are linearly dependent. Since p_i could not coincide with the area center by the definition, the condition (1.1) says that there exists a constant a_i such that

$$p_{i-1} + p_{i+1} = a_i p_i \tag{1.2}$$

for every i. Here we note that the equality (1.2) is expressed as

$$\begin{pmatrix} p_{i-1} & p_i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_i \end{pmatrix} = \begin{pmatrix} p_i & p_{i+1} \end{pmatrix}. \tag{1.3}$$

Hence if we denote the matrix $\begin{pmatrix} 0 & -1 \\ 1 & a_i \end{pmatrix}$ by $A(a_i)$, we see that the equality

$$\begin{pmatrix} p_0 & p_1 \end{pmatrix} A(a_1)A(a_2)\cdots A(a_n) = \begin{pmatrix} p_0 & p_1 \end{pmatrix}. \tag{1.4}$$

holds. Since p_0 and p_1 are linearly independent by the definition, the equality (1.4) implies that

$$A(a_1)A(a_2)\cdots A(a_n) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.5}$$

Conversely if the condition (1.5) is met, then the points $p_0, \cdots, p_{n-1}$ consitute the vertices of an n-gon with the origin as an area-center. Furthermore it is an unexpected coincidence that the product on the left hand side of (1.5) plays an important role in our previous study of the Chebyshev varieties in [2], [3]. Let us recall some of definitions introduced there. For n independent variables $x_1, \cdots, x_n$, let

$$U(x_1, \cdots, x_n) = \begin{pmatrix} x_1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & x_2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & x_3 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & x_{n-1} & -1 \\ 0 & 0 & 0 & \cdots & -1 & x_n \end{pmatrix},$$

and let

$$u(x_1, \cdots, x_n) = \det U(x_1, \cdots, x_n).$$

The zero locus $V(u)$ of u in $\mathbb{A}^n$ is called the *Chebyshev variety* (of the second kind), and is denoted simply by V_n. For simplicity we use the notation $u[i, j]$ to express $u(x_i, x_{i+1}, \cdots, x_j)$ when $i < j$, with the convention that

$$u[i, i] = x_i, u[i, i-1] = 1,$$

for any $i \geq 1$. We recall some identities proved in [2], which will be used throughout the paper. The first one is a recurrence relation

$$u[1, n] = x_1 u[2, n] - u[3, n], \tag{1.6}$$

which holds for any integer $n \geq 2$. By symmetry, for any $n \geq 2$, we have

$$u[1, n] = x_n u[1, n-1] - u[1, n-2]. \tag{1.7}$$

The following identity, which holds for any variables x, y, will play a crucial role when we employ an inductive argument in this paper:

$$A(x)A(0)A(y) = -A(x + y). \tag{1.8}$$

Of fundamental importance for our study is the identity:

$$A(x_n)A(x_{n-1}) \cdots A(x_1) = \begin{pmatrix} -u[2, n-1] & -u[1, n-1] \\ u[2, n] & u[1, n] \end{pmatrix}. \tag{1.9}$$

Comparing the order of multiplications on the left hand side of (1.9) with that of (1.5), we need to know what occurs if we reverse the order of multiplications on the left hand side:

Lemma 1.1.

$$A(x_1)A(x_2) \cdots A(x_n) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} {}^t(A(x_n)A(x_{n-1}) \cdots A(x_1)) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. This is a direct consequence of the identity

$$ {}^{t}A(x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A(x) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad \square $$

It follows from this lemma and the equality (1.9) that

$$ A(x_1)A(x_2)\cdots A(x_n) = \begin{pmatrix} -u[2,n-1] & -u[2,n] \\ u[1,n-1] & u[1,n] \end{pmatrix}. \tag{1.10} $$

Here we record the following polynomial identity which follows from (1.8) and (1.10):

Lemma 1.2. *For any positive integer n for any k with $2 \leq k \leq n-1$, we have*

$$ \begin{aligned} & u(x_1, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_n) \\ & = -u(x_1, \cdots, x_{k-2}, x_{k-1} + x_{k+1}, x_{k+2}, \cdots, x_n). \end{aligned} $$

Now combining (1.5) with (1.10), we obtain the following:

Proposition 1.2. *An n-gon in $\mathbb{R}^2$ has the origin as its area-center if and only if there exists a real solution of the simultaneous equation*

$$ u[1,n] = 1, \tag{1.11} $$

$$ u[1,n-1] = 0, \tag{1.12} $$

$$ u[2,n] = 0. \tag{1.13} $$

Remark 1.2. Note that the first equality (1.11) can be replaced by the equality

$$ u[2,n-1] = -1, \tag{1.14} $$

since $\det A(x) = 1$ as a polynomial in x.

Definition 1.2. *We call the subvariety of $\mathbb{A}^n$ defined by the three equations in Proposition* 1.2 *the AC-variety and denote it by AC_n.*

We illustrate Proposition 1.2 by a few examples.

Example 1.1. When $n = 3$, the three equations (1.11), (1.12), (1.13) becomes

$$ \begin{cases} u[1,3] = x_1x_2x_3 - x_1 - x_3 = 1, \\ u[1,2] = x_1x_2 - 1 = 0, \\ u[2,3] = x_2x_3 - 1 = 0. \end{cases} $$

Inserting $x_1x_2 = 1$, which is implied by the second equation, into the first equation, we have $x_1 = -1$, and it follows that $x_2 = x_3 = -1$. Hence the equalities (1.2) for $i = 1, 2, 3$ require one and the same equation $p_0 + p_1 + p_2 = 0$, which means that the origin is the barycenter, as is seen before.

Example 1.2. When $n = 4$, recalling that the equation (1.11) can be replaced by (1.14) (see Remark 1.2), we see that the defining equations of AC_4 become

$$\begin{cases} u[2,3] = x_2x_3 - 1 = -1, \\ u[1,3] = x_1x_2x_3 - x_1 - x_3 = 0, \\ u[2,4] = x_2x_3x_4 - x_2 - x_4 = 0. \end{cases} \tag{1.15}$$

The first equality in (1.16) implies that x_2 or x_3 is equal to zero. When $x_2 = 0$, it follows from the second and the third equalities that $x_1 + x_3 = x_4 = 0$, and when $x_3 = 0$, we have $x_1 = x_2 + x_4 = 0$. In the case of $x_2 = 0$, therefore, the four conditions for $i = 1, 2, 3, 4$ of (1.2) become

$$\begin{cases} p_0 + p_2 = x_1p_1, \\ p_1 + p_3 = 0, \\ p_2 + p_0 = -x_1p_3, \\ p_3 + p_1 = 0. \end{cases} \tag{1.16}$$

Note that the third (resp. the fourth) equation is equivalent to the first (resp. the second) equation, hence (1.16) is simplified to

$$\begin{cases} p_0 + p_2 = x_1p_1, \\ p_1 + p_3 = 0. \end{cases} \tag{1.17}$$

Similarly in the case of $x_3 = 0$, the conditions are reduced to

$$\begin{cases} p_0 + p_2 = 0, \\ p_1 + p_3 = x_2p_2, \end{cases} \tag{1.18}$$

Hence we see that a quadrilateral $P = (p_0, p_1, p_2, p_3)$ has an area-center (not necessarily equal to the origin) if and only if

$$\text{the midpoint of the segment } p_0p_2 \text{ lies on the line } p_1p_3, \tag{1.19}$$

or

$$\text{the midpoint of the segment } p_1p_3 \text{ lies on the line } p_0p_2. \tag{1.20}$$

Using these criteria (1.19), (1.20), we can show that there is a quadrilateral *without* area-center:

Proposition 1.3. *Let $p_0 = (0,1), p_1 = (-1,0), p_2 = (0,-1) \in \mathbb{R}^2$. For a point $p_3 \in \mathbb{R}^2$ to constitute a quadrilateral $P = (p_0, p_1, p_2, p_3)$ with area-center, it is necessary and sufficient that $p_3 \in V(x-1) \cup (V(y) \setminus \{p_1\}))$.*

Proof of Proposition 1.3. If the condition (1.19) is satisfied, then the origin must lies on the line p_1p_3, hence the point p_3 must lie on x-axis $V(y)$ and cannot coincides with p_1 by Definition 1.1. If the condition (1.20) is satisfied, then the point p_3 must lie on the line $V(x-1)$. □

It follows that an n-gon with $n \geq 4$ does not always have an area-center. This fact motivates us to investigate what kind of polygons have area-centers.

2 Geometry of the AC-varieties

Since the equations (1.11)-(1.13) of AC_n can be defined over any field, we will fix an arbitrary field K and regard AC_n is defined over K from now on. In this section we focus on some of algebro-geometric properties of the AC-varieties.

Firstly we construct a Groebner basis of its defining ideal:

Proposition 2.1. *For any integer $n \geq 4$, let $I = \langle u[1,n]-1, u[2,n], u[1,n-1]\rangle$, the defining ideal of AC_n. With respect to the lexicographic order with $x_1 > x_2 > \cdots > x_n$, a Groebner basis of I is given by*

$$I = \langle u[3,n]+1, u[4,n]+x_2, u[3,n-1]+x_1\rangle. \tag{2.1}$$

Proof. Let J denote the ideal on the right hand side of (2.6). First we show that $I = J$.
$J \subset I$: As for the first generator $u[3,n]+1$ of J, since $u[1,n] = x_1u[2,n]-u[3,n]$ holds by (1.6), we have

$$u[3,n]+1 = x_1u[2,n]-(u[1,n]-1) \in I$$

This implies in turn that

$$u[2,n] = x_2u[3,n]-u[4,n] \equiv -x_2-u[4,n] \pmod{I},$$

hence the second generator $u[4,n]+x_2$ of J belongs to I. As for the third generator, we employ the equality (1.10). Taking the determinants of both sides of (1.10) and noting that $\det A(x) = 1$, we have

$$-u[1,n]u[2,n-1]+u[1,n-1]u[2,n] = 1. \tag{2.2}$$

Since $u[1,n] \equiv 1 \pmod{I}, u[2,n] \equiv 0 \pmod{I}$, it follows that

$$-u[2,n-1] \equiv 1 \pmod{I},$$

which implies that

$$u[2,n-1]+1 \in I.$$

Hence we have $u[1,n-1] = x_1u[2,n-1]-u[3,n-1] \equiv -x_1-u[3,n-1] \pmod{I}$. Since $u[1,n-1] \in I$, it follows that $u[3,n-1]+x_1 \in I$. Thus we see that $J \subset I$.
$I \subset J$: For the second generator $u[2,n]$ of I, the equality

$$u[2,n] = x_2u[3,n]-u[4,n] = x_2(u[3,n]+1)-(u[4,n]+x_2)$$

implies that $u[2,n] \in J$. Therefore we have $u[1,n]-1 = x_1u[2,n]-(u[3,n]+1) \in J$. Then the equality (2.7) modulo J this time implies that

$$-u[2,n-1] \equiv 1 \pmod{J}.$$

hence we have $u[2, n-1] + 1 \in J$. It follows that

$$u[1, n-1] = x_1(u[2, n-1] + 1) - (u[3, n-1] + x_1) \in J.$$

Thus all of the generators of I belong to J, and we see that $I = J$. What remains is to show that the three generators actually gives us a Groebner basis. Note that the leading monomials of them are

$$\begin{aligned} \mathrm{LM}(u[3, n] + 1) &= x_3 x_4 \cdots x_n, \\ \mathrm{LM}(u[4, n] + x_2) &= x_2, \\ \mathrm{LM}(u[3, n-1] + x_1) &= x_1. \end{aligned}$$

Since these are pairwise relatively prime, it follows from Buchberger's simplified criterion (see [1, Chapter 2, Section 9, Theorem 3, and Proposition 4], for example) that the three polynomials on the right hand side of (2.1) constitute a Groebner basis of I. This completes the proof of Proposition 2.1. □

As a direct consequence, we have the following:

Corollary 2.1. *The AC-variety AC_n is a complete intersection of dimension $n-3$ in $\mathbb{A}^n$.*

In order to find when AC_N is nonsingular, we first investige the singular points on the hypersurface $V(u[1, n] + 1) \subset \mathbb{A}^n$ related to the first element of the Groebner basis (2.1) of the defining ideal. We need the following simple lemma:

Lemma 2.1. *For any integer k with $1 \le k \le n$, we have*

$$\frac{\partial}{\partial x_k} u[1, n] = u[1, k-1] u[k+1, n].$$

Proof. This is a direct consequence of the definition $u[1, n] = \det U(x_1, \cdots, x_n)$ and the rule for the differentiation of a deteminant. □

Proposition 2.2. (1) *If $n \not\equiv 2 \pmod 4$, then $V(u[1, n] + 1)$ is nonsingular.*
(2) *If $n \equiv 2 \pmod 4$, then $V(u[1, n] + 1)$ has the origin as its unique singular point.*

Proof. It follows from Lemma 2.1 that a singular point of $V(u[1, n] + 1)$ satisfies the simultaneous equation

$$\begin{cases} u[2, n] = 0, \\ u[1, 1] u[3, n] = 0, \\ u[1, 2] u[4, n] = 0, \\ \cdots, \\ u[1, n-2] u[n, n] = 0, \\ u[1, n-1] = 0 \end{cases} \tag{2.3}$$

Here we employ the following simple lemma, which facilitates our proof greatly:

Lemma 2.2. (1) *For any k with $2 \le k \le n$, we have $V(u[1,k]) \cap V(u[1,k-1]) = \phi$.*
(2) *For any k with $2 \le k \le n$, we have $V(u[k,n]) \cap V(u[k-1,n]) = \phi$.*

Proof of Lemma. (1) Contrary to the conclusion, suppose that there exists a point $(x_i) \in \mathbb{A}^n$ such that

$$(x_l) \in V(u[1,k]) \cap V(u[1,k-1]). \tag{2.4}$$

Since $u[1,k] = x_k u[1,k-1] - u[1,k-2]$ holds by (1.7), the assumption (2.4) implies that $(x_i) \in V(u[1,k-2])$, if $k \ge 3$. This leads us eventually to the conclusion

$$(x_i) \in V(u[1,2]) \cap V(u[1,1])$$

Recalling that $u[1,2] = x_1x_2 - 1, u[1,1] = x_1$, we have $V(u[1,2]) \cap V(u[1,1]) = \phi$. This contradiction completes the proof of (1). The claim (2) can be proved similarly, by appealing to the fact that $V(u[n-1,n]) \cap V(u[n,n]) = \phi$. □

Now we resume the proof of Proposition 2.3. It follows from this lemma that the first equation of (2.3) forces us to choose the first alternative $u[1,1] = 0$ of the consequence "$u[1,1] = 0$ or $u[3,n] = 0$" of the second equation of (2.3). Which in turn forces us to choose the alternative $u[4,n] = 0$ of the third equation of (2.3). Repeating similarly, we find that

$$\text{the number of equation } n \text{ in (2.3) must be necessarily even.} \tag{2.5}$$

From here on, we assume that the condition (2.5) is met. Then it follows from (2.3) and Lemma 2.2 that

$$u[2,n] = u[4,n] = \cdots = u[n,n] = 0, \tag{2.6}$$
$$u[1,1] = u[1,3] = \cdots = u[1,n-1] = 0. \tag{2.7}$$

Since we have $u[2,n] = x_2u[3,n] - u[4,n]$ and $u[3,n] \ne 0$ by our lemma, the first equality in (2.6) implies that $x_2 = 0$. Repeating this argument with the remaining equalities in (2.6), we must have

$$x_2 = x_4 = \cdots = x_n = 0.$$

Similaryly it follows from the equalities (2.7) that

$$x_1 = x_3 = \cdots = x_{n-1} = 0.$$

Thus we are reduced to checking when the origin of $\mathbb{A}^n$ with n even belongs to $V(u[1,n]+1)$. Since $A(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $A(0)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, the (2,2)-entry $u[1,n]$ of the matrix on the right hand side of (1.10) takes the value -1 at the origin if and only if $n \equiv 2 \pmod 4$. This completes the proof of Proposition 2.3. □

Now we can show the following:

Theorem 2.1. *For any integer $n \not\equiv 0 \pmod 4$, the AC-variety AC_n is a nonsingular complete intersection. When $n \equiv 0 \pmod 4$, the AC-variety AC_n has the origin as its unique singular point.*

Proof. Let

$$
\begin{aligned}
f_1 &= x_1 + u[3, n-1], &(2.8)\\
f_2 &= x_2 + u[4, n], &(2.9)\\
f_3 &= u[3, n] + 1, &(2.10)
\end{aligned}
$$

which constitute a Groebner basis of the defining ideal of AC_n by Proposition 2.1. Furthermore let $M = (m_{ij})_{1 \le i \le 3, 1 \le j \le n}$ denote the 3-by-n matrix with $m_{ij} = \partial f_i / \partial x_j$. Then we see that the 3-by-2 submatrix $(m_{ij})_{1 \le i \le 3, 1 \le j \le 2}$ of M equals to

$$
\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.
$$

Hence the set of singular points on AC_n is contained in that of the zero locus $V(f_3)$. We know, however, by Proposition 2.3 that the latter is empty when $n \not\equiv 0 \pmod 4$. (Note that $f_3 = u[3, n] + 1$ and the proposition looks at its zero locus in $\mathbb{A}^{n-2}$.) Hence the first assertion of the theorem follows. Thus we are reduced to considering the case when $n \equiv 0 \pmod 4$. in this case, Proposition 2.3 tells us that $m_{3j} = 0$ $(1 \le j \le n)$ only when $(x_3, \cdots, x_n) = (0, \cdots, 0)$. Furthermore if $(x_3, \cdots, x_n) = (0, \cdots, 0)$, then it follows from (1.6) that $u[4, n] = -u[6, n] = \cdots = u[n, n] = x_n = 0$, and $u[3, n-1] = 0$ for a similar reason. Hence (2.8) and (2.9) implies that if $(x_i) \in AC_n$ and $(x_3, \cdots, x_n) = (0, \cdots, 0)$, then $x_1 = x_2 = 0$ too. This completes the proof. □

Next we show that the AC-varieties are rational:

Proposition 2.3. *For any integer $n \ge 5$, the AC-variety AC_n is rational.*

Remark 2.1. *The case when $n = 4$ is considered in Example* 1.2 *already, and we know that AC_4 is the union of two lines $V(x_1, x_3, x_2 + x_4)$ and $V(x_2, x_4, x_1 + x_3)$ in $\mathbb{A}^4$.*

Proof. Let $x = (x_1, \cdots, x_n)$ be a general point on AC_n. Since the equality

$$
u[3, n] = x_3 u[4, n] - u[5, n]
$$

holds for any $n \ge 5$ and $u[3, n] = -1$ by (2.10), we have

$$
x_3 = \frac{u[5, n] - 1}{u[4, n]}. \tag{2.11}
$$

The equality (2.9) implies directly that

$$
x_2 = -u[4, n].
$$

Furthermore it follows from (2.8) that

$$x_1 = -u[3, n-1] = -x_3 u[4, n-1] + u[5, n-1].$$

Inserting the right hand side of (2.11) into this expression, we have

$$\begin{aligned} x_1 &= -\left(\frac{u[5,n]-1}{u[4,n]}\right) u[4, n-1] + u[5, n-1] \\ &= \frac{-(u[4,n-1]u[5,n] - u[4,n]u[5,n-1]) + u[4,n-1]}{u[4,n]}. \end{aligned}$$

Amazingly enough, the expression in the bracket in the numerator of the last fraction is equal to

$$\det(A(x_4)A(x_5)\cdots A(x_n))$$

by (1.10), hence it is equal to one and we have

$$x_1 = \frac{u[4, n-1] - 1}{u[4, n]}.$$

This completes the proof of the rationliaty. □

3 Brackets

In [3] we see how a string of brackets corresponds to a subvariety of the Chebyshev varieties. We will generalize the correspondence in various ways in this paper.

A string of left brackets and right brackets is said to be balanced if each left bracket has a matching right bracket. To be more precise, we define a function $s : \{(,)\} \to \{\pm 1\}$ by $s(\text{"("}) = 1, s(\text{")"}) = -1$. Then the balancedness is defined as follows:

Definition 3.1. *A string $(b_i)_{1\leq i\leq 2n}$ is balanced if and only if $\sum_{1\leq i\leq 2n} s(b_i) = 0$ and $\sum_{1\leq i\leq k} s(b_i) \geq 0$ for any k with $1 \leq k \leq 2n$.*

For a positive integer n, let Par_n ("Par" for parentheses) denote the set of balanced strings of brackets of length $2n$. For a balanced string $b = (b_1, \cdots, b_{2n}) \in Par_n$, let $Pairs(b)$ denote the set of matching pair of left and right brackets. On the set $Pairs(b)$ we introduce a partial order " $\prec$ ": For $(b_i, b_j), (b_k, b_\ell) \in Pairs(b)$, $(b_i, b_j) \prec (b_k, b_\ell)$ if and only if $k < i$ and $j < \ell$. We denote by $L(b)$ (resp. $R(b)$) the set of left (resp. right) brackets among $\{b_i\}$. When b_i belongs to $L(b)$, its matching right bracket will be denoted by $r(b_i) \in R(b)$. Furthermore for any $b_i \in L(b)$ let $ls(b_i)$ denote the set of left brackets in $\{b_{i+1}, \cdots, r(b_i)\}$. Note that $ls(b_i) = \phi$, when $b_{i+1} \in R(b)$. Association of a left bracket b_i with the matching pair $(b_i, r(b_i))$ defines a natural bijection from the set of left brackets $L(b)$ to the set of matching pairs $Pairs(b)$. Through this correspondence we translate the partial order " $\prec$ " on $Pairs(b)$ into that on $L(b)$, which will be denoted by the same symbol " $\prec$ ". Then the poset $L(b)$ endowed with this partial order becomes a ranked poset with the rank function *rank* defined as follows:

Definition 3.2. *For a balanced bracket $b = (b_1, \cdots, b_{2n}) \in Par_n$ and for any left bracket b_i in b, the rank of b_i, denoted by $rank(b_i)$, is defined inductively by the rule*

$$\begin{cases} rank(b_i) = 1, & if\, r(b_i) = b_{i+1}, \\ rank(b_i) = \max_{b_k \in ls(b_i)} rank(b_k) + 1, & otherwise. \end{cases}$$

Moreover the maximum of the ranks of the left brackets in $L(b)$ is called the height *of b and is denoted by $height(b)$. Furthermore for any $k \in [1, height(b)]$ we set*

$$ls_{rank=k}(b) = \{b_i \in L(b); rank(b_i) = k\}.$$

In order to associate a set of polynomials with a balanced string, we regard a string $b = (b_1, \cdots, b_{2n}) \in Par_n$ is situated along the number line such that b_i is located at $i - 0.5$ for every $i \in \{1, \cdots, 2n\}$. Its coordinate will be denoted by $pos(b_i)$ so that $pos(b_i) = i - 0.5$. For any pair of real numbers p, q with $p < q$, we put $[p, q] = \{k \in \mathbb{Z}; p \leq k \leq q\}$, and for any i, j with $1 \leq i < j \leq 2n$, we put $[b_i, b_j] = [pos(b_i), pos(b_j)]$. Most important notion for a string in Par_n is that of *content*. It is defined as follows. Let b_i be a left bracket. Then the content $cont(b_i)$ is defined by the following rule:

$$cont(b_i) = [b_i, r(b_i)] \setminus \bigcup_{b_k \in ls(b_i)} [b_k, r(b_k)].$$

Furtheremore collecting all contents of $b_i \in L(b)$, we set

$$\begin{aligned} cont(b) &= \bigcup_{b_i \in L(b)} \{cont(b_i)\}, \\ num(b) &= \bigcup_{b_i \in L(b)} cont(b_i). \end{aligned}$$

Note here there is a slight distinction between $cont(b)$ and $num(b)$. Let us illustrate the notation introduced above by some examples.

Example 1.1
Let $b = (b_1, \cdots, b_6) =$ "(()())" $\in Par_3$. By the rule mentioned above, it is expressed on the number line as follows:

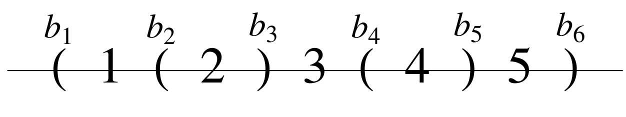

Fig.1 String $(()())$ on the number line with its contents

We see that $L(b) = \{b_1, b_2, b_4\}$ and $R(b) = \{b_3, b_5, b_6\}$. The matching right brackets are given by $r(b_1) = b_6, r(b_2) = b_3$, and $r(b_4) = b_5$, hence the set of matching pairs is given by

$$Pairs(b) = \{(b_1, b_6), (b_2, b_3), (b_4, b_5)\}.$$

The pairs are ordered as

$$(b_2, b_3) \prec (b_1, b_6) \succ (b_4, b_5),$$

and (b_2, b_3) and (b_4, b_5) are not comparable. Hence the Hasse diagram for the poset $Pairs(b)$ is depicted as follows:

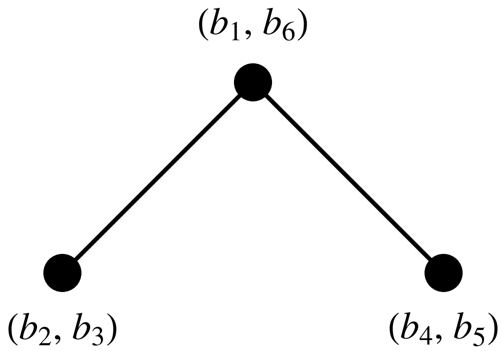

Fig.2 Hasse diagram for $Pairs(b)$

The rank of (b_2, b_3) and (b_4, b_5) are equal to 1, and that of (b_1, b_6) is 2. The contents are computed as follows;

$$\begin{aligned} cont(b_1) &= [b_1, b_6] - ([b_2, b_3] \cup [b_4, b_5]) \\ &= \{1, 2, 3, 4, 5\} - (\{2\} \cup \{4\}) \\ &= \{1, 3, 5\}, \\ cont(b_2) &= \{2\}, \\ cont(b_4) &= \{4\}. \end{aligned}$$

Therefore we have

$$\begin{aligned} cont(b) &= \{\{1, 3, 5\}, \{2\}, \{4\}\}, \\ num(b) &= \{1, 2, 3, 4, 5\}. \end{aligned}$$

Example 1.2

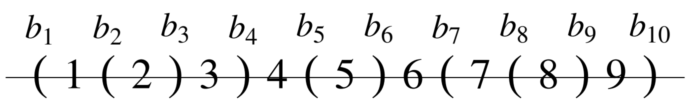

Fig.3 String (())()(()) on the number line with its contents

This time the Hasse diagram for $Pairs(b)$ is depicted as follows:

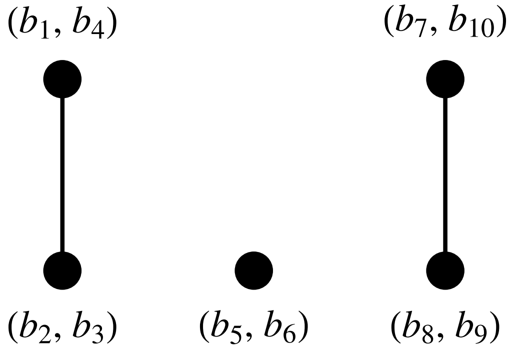

Fig.4 Hasse diagram for $Pairs(b)$

Furthermore we have

$$\begin{aligned} cont(b) &= \{\{1,3\},\{7,9\},\{2\},\{5\},\{8\}\}, \\ num(b) &= \{1,2,3,5,7,8,9\}. \end{aligned}$$

In these examples we observe that each $cont(b_i)$ for $b_i \in L(b)$ consists either wholly of odd integers or wholly of even integers. The following proposition shows that this observation is correct:

Proposition 3.1. *For any string $b \in Par_n$ and for any $b_i \in L(b)$, the parities of elements in $cont(b_i)$ are one and the same.*

Proof. We prove this by induction on the rank of $b_i \in L(b)$. When the rank is equal to one, there is nothing to prove since $cont(b_i) = \{i\}$ in this case. Therefore suppose that $rank(b_i) = r \geq 2$ and the assertion is proved for every left bracket of smaller rank. Let us put $ls(b_i) \cap ls_{rank=r-1}(b) = \{b_{i_1}, \cdots, b_{i_p}\}$. Note that $i_1 = i+1$, since $rank(b_i) = 1$ if otherwise. Assume that i is odd. Then we have $\min(cont(b_{i_1})) = \min(cont(b_{i+1})) = i+1 \equiv 0 \pmod 2$, and hence $\max(cont(b_{i_1})) \equiv 0 \pmod 2$ by the induction hypothesis. It follows that the second smallest element of $cont(b_i)$ is equal to $\max(cont(b_{i_1})) + 1$, and hence it is odd, which in turn shows that $\min(cont(b_{i_2})) = \max(cont(b_{i_1})) + 2 \equiv 0 \pmod 2$, and hence the third smallest element of $cont(b_i)$ is odd, and so on. Arguing in this way we arrive at the conclusion that every element in $cont(b_i)$ is odd. Since the case when i is even can be treated similarly, we finish the proof. □

Now we attach a set of linear polynomials to each balanced string of brackets. Fix a nonnegative integer n and let $x_1, \cdots x_{2n}$ be independent variables. For any subset $S \subset [1, 2n]$, let f_S denote the linear polynomial $\sum_{k \in S} x_k$. When

$b \in Par_n$, we define a set of polynomial f_b as

$$f_b = \{f_{cont(b_i)}; b_i \in L(b)\}.$$

Therefore for the string b in Example 1.1, we have

$$f_b = \{x_1 + x_3 + x_5, x_2, x_4\},$$

and for b in Example 1.2, we have

$$f_b = \{x_1 + x_3, x_7 + x_9, x_2, x_5, x_8\}.$$

We will see later that the zero locus $V(f_b)$ for every $b \in Par_n$ provides us with a subvariety of AC_n. The following proposition plays a crucial role several times when we employ an inductive argument to show this fact and some related results:

Proposition 3.2. *Let $b = (b_i) \in Par_n$ and suppose that there exists a minimal element $b_k \in L(b)$ which is not maximal. Let $b' \in Par_{n-1}$ denote the string $(b_1, \cdots, b_{k-1}, b_{k+2}, \cdots, b_{2n})$. Let $p_k : \mathbb{A}^{2n-1} \to \mathbb{A}^{2n-3}$ denote the linear map defined by the rule*

$$p_k(x_1, \cdots, x_{2n-1}) = (x_1, \cdots, x_{k-2}, x_{k-1} + x_{k+1}, x_{k+2}, \cdots, x_{2n-1}).$$

Then we have

$$p_k^*(f_{b'}) = f_b \setminus \{x_k\}. \tag{3.1}$$

Before we begin our proof, we illustrate the strings b and b' when $b = (())(()()) \in Par_5$ and we choose b_6 as a minimal but non-maximal element in $L(b)$. The upper cover of b_6 is b_5 in this example:

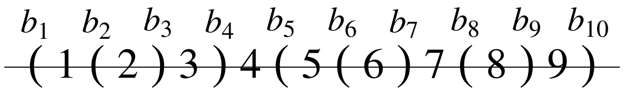

Fig.5 String b =(())(()()) with its contents

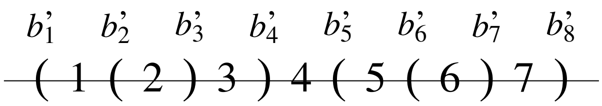

Fig.6 String b' with its contents

The content $cont(b)$ and the associated set of polynomials f_b are given by

$$cont(b) = \{cont(b_1), cont(b_2), cont(b_5), cont(b_6), cont(b_8)\},$$

with

$$\begin{aligned} cont(b_1) &= \{1,3\}, \\ cont(b_2) &= \{2\}, \\ cont(b_5) &= \{5,7,9\}, \\ cont(b_6) &= \{6\}, \\ cont(b_8) &= \{8\}, \end{aligned}$$

and

$$f_b = \{x_1 + x_3, x_2, x_5 + x_7 + x_9, x_6, x_8\}.$$

As for the string b', we have

$$cont(b') = \{cont(b'_1), cont(b'_2), cont(b'_5), cont(b'_6)\},$$

with

$$\begin{aligned} cont(b'_1) &= \{1,3\}, \\ cont(b'_2) &= \{2\}, \\ cont(b'_5) &= \{5,7\}, \\ cont(b'_6) &= \{6\}. \end{aligned}$$

and

$$f_{b'} = \{y_1 + y_3, y_2, y_5 + y_7, y_6\}.$$

The map $p_6 : \mathbb{A}^9 \to \mathbb{A}^7$ is given by

$$p_6(x_1, \cdots, x_9) = (y_1, \cdots, y_7)$$

with

$$y_1 = x_1, \cdots, y_4 = x_4, y_5 = x_5 + x_7, y_6 = x_8, y_7 = x_9.$$

Therefore we have

$$\begin{aligned} p_6^*(f_{b'}) &= \{x_1 + x_3, x_2, x_5 + x_7 + x_9, x_8\} \\ &= f_b \setminus \{x_6\}, \end{aligned}$$

which shows that Proposition 3.2 holds for this $b \in Par_5$. This example will facilitate the reader to follow the argument in the proof.

Proof. Let $b_\ell \in L(b)$ be the upper cover of b_k, which is unique by the definition of the partial order on $L(b)$. It follows from the definition of the content that

$$\begin{aligned} cont(b_k) &= \{k\}, \\ cont(b_\ell) &\ni k-1, k+1, \end{aligned}$$

and that

$$cont(b_m) \not\ni k-1, k, k+1, \tag{3.2}$$

for any other $b_m \in L(b)\backslash\{b_k, b_\ell\}$, since the contents for distinct left brackets have no element in common. By the definition of p_k, if we employ the coordinates $(x_i)_{1\le i\le 2n-1}$ for the source $\mathbb{A}^{2n-1}$ and $(y_j)_{1\le j\le 2n-3}$ for the target $\mathbb{A}^{2n-3}$, then $p_k((x_i)) = (y_j)$ with

$$y_j = \begin{cases} x_j, & j \le k-2, \\ x_{k-1} + x_{k+1}, & j = k-1, \\ x_{j+2}, & j \ge k. \end{cases} \tag{3.3}$$

We need to know the difference between the content of b and that of b', and will see how the associated sets of polynomials f_b and $f_{b'}$ are related through the pull-back by the linear map p_k. For this pupose we introduce a pair of monotone maps $minus2_k : [1, 2n-1] \to [1, 2n-3]$, $plus2_k : [1, 2n-3] \to [1, 2n-1]$, defined by the following rules:

$$\begin{aligned} minus2_k(i) &= \begin{cases} i, & i \le k-1, \\ k-1, & i = k, \\ i-2, & i \ge k+1, \end{cases} \\ plus2_k(i) &= \begin{cases} i, & i \le k-1, \\ i+2, & i \ge k. \end{cases} \end{aligned}$$

Note here that the pair satisfies the identity

$$(plus2_k \circ minus2_k)(i) = i, \quad for\ i \ne k, k+1. \tag{3.4}$$

With the help of the map $minus2_k$, we see that the content of b' is expressed as follows:

$$cont(b') = \{minus2_k(cont(b_j)); b_j \in L(b) - \{b_k\}\}.$$

On the other hand if $b_m \in L(b) \setminus \{b_k, b_\ell\}$, then $k-1, k, k+1 \notin cont(b_m)$ by (3.2), hence it follows from (3.3) and (3.4) that

$$\begin{aligned} p_k^*(f_{minus2_k(cont(b_m))}) &= p_k^*(\sum_{j\in minus2_k(cont(b_m))} y_j) \\ &= \sum_{j\in minus2_k(cont(b_m))} x_{plus2_k(j)} \\ &= \sum_{i\in cont(b_m)} x_i \\ &= f_{b_m} \end{aligned}$$

For b_ℓ, recall that $k-1, k+1 \in cont(b_\ell)$, and hence $k, k+2 \notin cont(b_\ell)$ by Proposition 3.1. Therefore, noting that $minus2_k(k-1) = minus2_k(k+1) = k-1$, we can compute the pull-back as follows:

$$\begin{aligned} p_k^*(f_{minus2_k(cont(b_\ell))}) &= p_k^*(\sum_{j\in minus2_k(cont(b_\ell))} y_j) \\ &= p_k^*(\sum_{j\in minus2_k(cont(b_\ell))\cap[1,k-2]} y_j) \\ &\quad +p_k^*(y_{k-1}) \\ &\quad +p_k^*(\sum_{j\in minus2_k(cont(b_\ell))\cap[k,2n-3]} y_j) \\ &= (\sum_{j\in minus2_k(cont(b_\ell))\cap[1,k-2]} x_j) \\ &\quad +(x_{k-1}+x_{k+1}) \\ &\quad +(\sum_{j\in minus2_k(cont(b_\ell))\cap[k,2n-3]} x_{j+2}) \\ &= \sum_{j\in cont(b_\ell)\cap[1,k-2]} x_j \\ &\quad +(x_{k-1}+x_{k+1}) \\ &\quad +(\sum_{j\in cont(b_\ell)\cap[k,2n-3]} x_j) \\ &= f_{b_\ell}. \end{aligned}$$

Thus we see that the equality (3.1) holds. This completes the proof. □

4 Strings with angle brackets

In this section we introduce a set of strings of length $2n$ each of which consists of $n-1$ pairs of round brackets and a pair of angle brackets $\langle\rangle$. Given such a string $b = (b_i)_{1\le i\le 2n}$, assume that $b_k = \langle$ and $b_\ell = \rangle$. Then the string b is said to be *balanced* if all of the three substrings $(b_i)_{1\le i\le k-1}, (b_i)_{k+1\le i\le \ell-1}$, and $(b_i)_{\ell+1\le i\le 2n}$ are balanced in the sense of Definition 3.1. Let Ang_n denote the set of balanced strings consisting of $n-1$ pairs of round brackets and a pair of angle brackets. Accordingly the set of left brackets $L(b)$ of $b \in Ang_n$ consists of $n-1$ left round brackets and one $\langle$ in b. A partial order on $L(b)$ is defined in the same way as for Par_n. Note that if $b \in Ang_n$ and $b_k = \langle$, then b_k is maximal by the definition of balancedness. For any $b \in Ang_n$. we also define the content $cont(b)$ in the same way as for Par_n with the only exception that when $b_k = \langle$, we express its content by using angle bracket. Let us illustrate this point by the following example.

Example 4.1. Let $b \in Ang_6$ be the string depicted as follows:

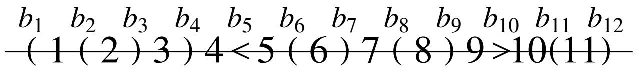

Fig.7 String $b = (())\langle()()\rangle()$ with its contents

The content is given by

$$cont(b) = \{\{2\}, \{6\}, \{8\}, \{11\}, \{1, 3\}, \langle 5, 7, 9\rangle\}.$$

Fix an element $c \in K$. For any $b \in Ang_n$,we assign a polynomial to every bracket in $L(b)$ by the following rule: If b_i is an round bracket, then f_{b_i} is defined to be the same as in the previous section, and if $b_i = \langle$, and $cont(b_i) = \langle k_1, \cdots, k_m\rangle$, then we define

$$f_{b_i}(c) = \sum_{1 \le i \le m} x_{k_i} - c.$$

Gathering these polynomials, we define the set of polynomials $f_b(c)$ by the rule

$$f_b(c) = \{f_{b_\ell}; b_\ell \in L(b) \setminus \{b_i\}\} \cup \{f_{b_i}(c)\}. \tag{4.1}$$

Therefore for the string b given in Example 4.1, we have

$$f_b(c) = \{x_2, x_6, x_8, x_{11}, x_1 + x_3, x_5 + x_7 + x_9 - c\},$$

The reason why we introduce this kind of the sets of polynomials is that they provide us with a family of linear subvarieties of Chebyshev varieties and their generalizations. For any $c \in K$ and for any pair (i, j) of positive integers with $i < j$, let

$$V_{[i,j]}(c) = \{(x_i, \cdots, x_j) \in \mathbb{A}^{j-i+1}; u[i, j] = c\}.$$

Theorem 4.1. *For any $c \in K$ and for any $b \in Ang_n$, we have*

$$V(f_b(c)) \subset V_{[1,2n-1]}((-1)^{n-1}c).$$

Proof. We prove this by induction on n. When $n = 1$, the set Ang_1 consists solely of $\langle\rangle$. If we call this string b, then

$$f_b(c) = \{x_1 - c\}.$$

On the other hand, we have

$$V_{[1,1]}((-1)^{1-1}c) = \{(x_1) \in \mathbb{A}^1; u[1, 1] = c\} = \{(x_1) \in \mathbb{A}^1; x_1 = c\},$$

hence the assertion is proved. When $n = 2$, the set Ang_2 has three strings:

$$Ang_2 = \{()\langle\rangle, \langle\rangle(), \langle()\rangle\}.$$

Let us call these strings b^1, b^2, b^3, respectively. Then it follows from the definition that

$$\begin{array}{rcl} f_{b^1}(c) &=& \{x_1, x_3 - c\}, \\ f_{b^2}(c) &=& \{x_1 - c, x_3\}, \\ f_{b^3}(c) &=& \{x_2, x_1 + x_3 - c\}. \end{array}$$

On the other hand, since $u[1,3] = x_1x_2x_3 - x_1 - x_3$, we have

$$V_{[1,3]}((-1)^{2-1}c) = \{(x_i) \in \mathbb{A}^3; x_1x_2x_3 - x_1 - x_3 = -c\},$$

and we see directly that the assertion holds true. Now assume that $n \geq 3$ and that the assertion is proved for any $n' < n$. Take an arbitrary balanced string $b = (b_1, \cdots, b_{2n}) \in Ang_n$. We divide our argument into two cases: (I) when every minimal element in $L(b)$ is maximal, and (II) when there exists an element in $L(b)$ which is minimal but not maximal. In another word, the Hasse diagram for b in the case (I) is totally disconnected and that in the case (II) is not so.

Case (I) Every minimal element in $L(b)$ is maximal. In this case we have $L(b) = \{b_1, b_3, \cdots, b_{2n-1}\}$ and that there is a unique integer k with $1 \leq k \leq n$ such that $b_{2k-1} = \langle$. It follows that

$$f_b(c) = \{x_1, x_3, \cdots, x_{2k-3}, x_{2k-1} - c, x_{2k+1}, \cdots, x_{2n-1}\}$$

On the other hand the equalities (1.6) and (1.7) imply that

$$\begin{array}{rl} & u(0, x_2, 0, x_4, \cdots, 0, x_{2k-2}, x_{2k-1}, x_{2k}, 0, x_{2k+2}, \cdots, 0, x_{2n-2}, 0) \\ = & (-1)^{k-1} \cdot (-1)^{n-k} \cdot u(x_{2k-1}) \\ = & (-1)^{n-1} x_{2k-1} \end{array}$$

Therefore if $a = (a_1, \cdots, a_{2n-1}) \in V(f_b(c))$, then we have

$$\begin{array}{rcl} u(a_1, \cdots, a_{2n-1}) &=& (-1)^{n-1} a_{2k-1} \\ &=& (-1)^{n-1} c. \end{array}$$

It follows that $a \in V_{[1,2n-1]}((-1)^{n-1}(c))$.

Case (II): There exists an element in $L(b)$ which is minimal but not maximal. Therefore we are in the situation treated in Proposition 3.2. As is assumed there, we call such an element b_k, and let b' denote the string defined by

$$b' = (b_1, \cdots, b_{k-1}, b_{k+2}, \cdots, b_{2n}).$$

Note that b_k is not angle bracket by the definition of balancedness, and hence b' belongs to Ang_{n-1}. Let $p_k : \mathbb{A}^{2n-1} \to \mathbb{A}^{2n-3}$ denote the map defined by

$$p_k(x_1, \cdots, x_{2n-1}) = (x_1, \cdots, x_{k-2}, x_{k-1} + x_{k+1}, x_{k+2}, \cdots, x_{2n-1}),$$

and let $\bar{p}_k$ denote its restriction to $V(x_k) \subset \mathbb{A}^{2n-1}$. Then it follows from Proposition that

$$p_k^*(f_{b'}(c)) = f_b(c) \setminus \{x_k\}.$$

Hence we have

$$p_k^{-1}(V(f_{b'}(c))) = V(f_b(c)). \tag{4.2}$$

Furthermore suppose that $a = (a_i) \in V(x_k)$ and $\bar{p}_k(a) \in V_{[1,2n-3]}((-1)^{n-2}c)$. Then it follows that

$$u(a_1, \cdots, a_{k-2}, a_{k-1} + a_{k+1}, a_{k+2}, \cdots, a_{2n-1}) = c.$$

The left hand side is equal to $-u(a_1, \cdots, a_{k-1}, 0, a_{k+1}, \cdots, a_{2k-1})$ by Lemma 1.2, and hence we have $u(a) = (-1)^{n-1}c$, and hence $a \in V_{[1,2n-1]}((-1)^{n-1}c)$. Thus we have

$$p_k^{-1}(V_{[1,2n-3]}((-1)^{n-2}c)) \quad \subset \quad V(x_k) \cap V_{[1,2n-1]}((-1)^{n-1}c). \tag{4.3}$$

On the other hand, by the induction hypothesis we have

$$V(f_{b'}(c)) \subset V_{[1,2n-3]}((-1)^{n-2}c),$$

hence (4.2) and (4.3) imply that

$$V(f_b(c)) \subset V(x_k) \cap V_{[1,2n-1]}((-1)^{n-1}c) \subset V_{[1,2n-1]}((-1)^{n-1}c).$$

This completes the proof for Case (II), hence at the same time the proof of Theorem 4.1. □

Note that when $c = 0$, the set $f_b(0)$ coincides with the set f_b introduced in the previous section. Hence, letting $c = 0$ in Theorem 4.1, we obtain the following corollary which is proved by a different and complicated method in [3]:

Corollary 4.1. *For any $b \in Par_n$, we have $V(f_b) \subset V_{2n-1}$.*

5 Strings with a bra-ket

Here *bra-ket* means the pair of *bra* $\langle|$ and *ket* $|\rangle$ as is introduced by Dirac. We regard a bra-ket as consisting of three elements "$\langle$" (the left angle bracket), "$|$" (the vertical bar), and "$\rangle$" (the right angle bracket). We define the balancedness of a string $b = (b_1, \cdots, b_{2n+1})$ with $n-1$ pairs of round brackets and one bra-ket as follows. Let $b_i = \langle, b_j = |, b_k = \rangle$ with $i < j < k$. Then b is said to be balanced if the four remaining substrings$(b_1, \cdots, b_{i-1}), (b_{i+1}, \cdots, b_{j-1}), (b_{j+1}, \cdots, b_{k-1})$, and $(b_{k+1}, \cdots, b_{2n+1})$ are balanced in the sense of Definition 3.1. We employ the convention that the left angle bracket b_i matches with the vertical bar b_j and the same vertical bar b_j matches with the right angle bracket b_k. Accordingly we regard a vertical bar both as a left bracket and as a right bracket, and we can define a partial order on the set $L(b)$ of left brackets in b in the same way as for Par_n. We denote the set of balanced strings of length $2n+1$ with $n-1$ pairs of round brackets and with one bra-ket by Bra_n. The content of a string $b \in Bra_n$ is defined in the same way as for Par_n with the only difference that the content of the bra-ket (b_i, b_j, b_k) is denoted by $\langle cont(b_i)|cont(b_j)\rangle$. The following example should clarify the meaning.

Example 5.1.

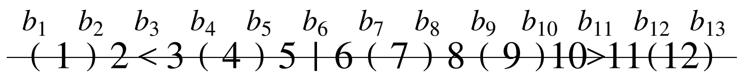

Fig.8 String $b = ()\langle()|()()\rangle()$ with its contents

The set $L(b)$ (resp. $R(b)$) of left (resp. right) brackets is given by

$$\begin{aligned} L(b) &= \{b_1, b_3, b_4, b_6, b_7, b_9, b_{12}\}, \\ R(b) &= \{b_2, b_5, b_6, b_8, b_{10}, b_{11}, b_{13}\}. \end{aligned}$$

The content of b is given by

$$cont(b) = \{\{1\}, \{4\}, \{7\}, \{9\}, \{12\}, \langle 3, 5|6, 8, 10\rangle\}.$$

For a string $b = (b_1, \cdots, b_{2n+1}) \in Bra_n$, let $b_i = \langle, b_j = |, b_k = \rangle$. For an arbitrary $c \in K$, we define the polynomial $f_{b_j}(c)$ by

$$f_{b_j}(c) = \sum_{p \in cont(b_i)} x_p \sum_{q \in cont(b_j)} x_q - c, \tag{5.1}$$

and for a left round bracket $b_\ell \in L(b)$, the polynomial f_{b_ℓ} is defined to be the same as in Section three. Gathering these polynomials, we define the set of polynomials $f_b(c)$ by the rule

$$f_b(c) = \{f_{b_\ell}; b_\ell \in L(b) \setminus \{b_i, b_j\}\} \cup \{f_{b_j}(c)\}. \tag{5.2}$$

Therefore for the string b in Example 5.1, we have

$$f_b(c) = \{x_1, x_4, x_7, x_9, x_{12}, (x_3 + x_5)(x_6 + x_8 + x_{10}) - c\}.$$

The reason why we introduce this third kind of polynomials is that the zero locus of $f_b(c)$ for $b \in Bra_n$ gives rise to a subvariety of degree two of the Chebyshev varieties. More precisely we can prove the following:

Theorem 5.1. *For any $b \in Bra_n$, we have $V(f_b(c)) \subset V_{[1,2n]}((-1)^{n-1}(c-1))$.*

Proof. We prove this by induction on n. When $n = 1$, the set Bra_1 consists solely of $\langle | \rangle$. If we call this string b, then its content is given by

$$cont(b) = \{\langle 1|2\rangle\}.$$

Therefore we have

$$f_b(c) = \{x_1x_2 - c\}$$

by (5.1) and (5.2). On the other hand, we have

$$\begin{aligned} V_{[1,2]}((-1)^{1-1}(c-1)) &= \{(x_1,x_2) \in \mathbb{A}^2; u[1,2] = c-1\} \\ &= \{(x_1,x_2) \in \mathbb{A}^2; x_1x_2 - 1 = c-1\} \\ &= \{(x_1,x_2) \in \mathbb{A}^2; x_1x_2 - c = 0\}, \end{aligned}$$

hence the assertion holds. When $n = 2$, the set Bra_2 has four strings:

$$Bra_2 = \{()\langle|\rangle, \langle|\rangle(), \langle()|\rangle, \langle|()\rangle\}.$$

Let us call these strings b^1, b^2, b^3, b^4, respectively. Then it follows from the definition that

$$\begin{aligned} f_{b^1}(c) &= \{x_1, x_3x_4 - c\}, \\ f_{b^2}(c) &= \{x_1x_2 - c, x_4\}, \\ f_{b^3}(c) &= \{x_2, (x_1+x_3)x_4 - c\}, \\ f_{b^4}(c) &= \{x_3, x_1(x_2+x_4) - c\}. \end{aligned}$$

Since $u[1,4] = x_1x_2x_3x_4 - x_1x_2 - x_3x_4 - x_1x_4 + 1$, we see directly that $V(f_{b^j}(c)) \subset V_{[1,4]}(1-c)$ $(1 \le j \le 4)$. Now assume that $n \ge 3$ and that the assertions is proved for any $n' < n$. Take an arbitrary balanced string $b = (b_1, \cdots, b_{2n+1}) \in Bra_n$. As is done in the proof of Theorem 4.1, we divide our argument into two cases: (I) when every minimal element in $L(b)$ is maximal, and (II) when there exists an element in $L(b)$ which is minimal but not maximal.

Case (I) Every minimal element in $L(b)$ is maximal. In this case the left angle bracket must be odd-indexed one b_{2k-1}, say, and hence we must have $b_{2k} = |$, and $b_{2k+1} = \rangle$. It follows that

$$\begin{aligned} &cont(b) \\ &= \{\{1\},\{3\},\cdots,\{2k-3\},\langle 2k-1|2k\rangle,\{2k+2\},\{2k+4\},\cdots,\{2n\}\}, \end{aligned}$$

hence we have

$$f_b(c) = \{x_1, x_3, \cdots, x_{2k-3}, x_{2k-1}x_{2k} - c, x_{2k+2}, x_{2k+4}, \cdots, x_{2n}\}.$$

On the other hand the identities (1.6) and (1.7) imply that

$$\begin{aligned} & u(0,x_2,0,x_4,\cdots,0,x_{2k-2},x_{2k-1},x_{2k},x_{2k+1},0,x_{2k+3},\cdots,0,x_{2n-1},0) \\ = \ & (-1)^{k-1} \cdot (-1)^{n-k} \cdot u(x_{2k-1},x_{2k}) \\ = \ & (-1)^{n-1}(x_{2k-1}x_{2k} - 1) \end{aligned}$$

Therefore if $a = (a_1, \cdots, a_{2n}) \in V(f_b(c))$, then we have

$$u(a_1, \cdots, a_{2n}) = (-1)^{n-1}(c-1),$$

and hence we have $a \in V_{[1,2n]}((-1)^{n-1}(c-1))$.

Case (II): There exists an element in $L(b)$ which is minimal but not maximal. Therefore we are in the situation treated in Proposition 3.2. As is done there, we call such an element b_k. Note that this is not a left angle bracket, since the latter is always maximal by the definition of balancedness. Let b' denote the string defined by

$$b' = (b_1, \cdots, b_{k-1}, b_{k+2}, \cdots, b_{2n}).$$

Note that b' belongs to Bra_{n-1}. Let $p : \mathbb{A}^{2n} \to \mathbb{A}^{2n-2}$ denote the map defined by

$$p(x_1, \cdots, x_{2n}) = (x_1, \cdots, x_{k-2}, x_{k-1} + x_{k+1}, x_{k+2}, \cdots, x_{2n}),$$

and let p_k denote its restriction to $V(x_k) \subset \mathbb{A}^{2n}$. Then by a similar reasoning to that employed in deduction of (4.2) and (4.3), we have

$$p_k^{-1}(V(f_{b'}(c))) = V(f_b(c)), \tag{5.3}$$
$$p_k^{-1}(V_{[1,2n-2]}((-1)^{n-2}(c-1)) \subset V(x_k) \cap V_{[1,2n]}((-1)^{n-1}(c-1)). \tag{5.4}$$

Since the induction hypothesis implies $V(f_{b'}(c)) \subset V_{[1,2n-2]}((-1)^{n-2}(c-1))$, it follows from (5.3) and (5.4) that

$$V(f_b(c)) \subset V(x_k) \cap V_{[1,2n]}((-1)^{n-1}(c-1)) \subset V_{[1,2n]}((-1)^{n-1}(c-1)).$$

This completes the proof for Case (II), hence at the same time the proof of Theorem 5.1. □

When $c = 1$, the variety $V_{[1,2n]}((-1)^{n-1}(c-1))$ coincides with the original Chebyshev variety V_{2n}, we have the following:

Corollary 5.1. *For any $b \in Bra_n$, we have $V(f_b(1)) \subset V_{2n}$.*

6 Associative transformation

In this section we investigate when the dimension of the intersection of subvarieties constructed in the previous sections becomes as large as posslble. This will enable us to construct a family of subvarieties of large dimension of AC-varieties. For this purpose we introduce some notation. Firstly we need *translation-by-one operator*. For any string $b \in Par_n$ and for any $b_i \in L(b)$, let

$$cont(b_i)^{(+1)} = \{k+1; k \in cont(b_i)\},$$

and let

$$cont(b)^{(+1)} = \{cont(b_i)^{(+1)}; b_i \in L(b)\}.$$

Accordingly we define the polynomial $f_{b_i}^{(+1)}$ for $b_i \in L(b)$ by

$$f_{b_i}^{(+1)} = \sum_{p \in cont b_i{}^{(+1)}} x_p,$$

and let

$$f_b^{(+1)} = \{f_{b_i}^{(+1)}; b_i \in L(b)\}.$$

Secondly we need the *associative transformation* which is defined as follows:

Definition 6.1. *For any $b \in Par_n$, suppose that $r(b_1) = b_i$. Let $b' = (b'_j)_{1 \leq j \leq 2n}$ be the string of bracket defined by*

$$b'_j = \begin{cases} b_{j+1}, & 1 \leq j \leq i-2, \\ (, & j = i-1 \\ b_{j+1}, & i \leq j \leq 2n-1, \\ ), & j = 2n. \end{cases}$$

The string b' is denoted by $ass(b)$, and is called the associative transformation *of b.*

Remark. The name comes from the usual associative law. If we take the equality $((a+b)+c)+d = (a+b)+(c+d)$ for example, which is legitimate by the associative law and forget the content, then both hand sides become $(())$ and $()()$. The latter is exactly the associative transformation of the former.

Example 6.1. For the string $b \in Par_5$ in Example 3.2,

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
(	(	)	)	(	)	(	(	)	)

Fig.**9** String $b = (())()(())$

the transformed string $b' = ass(b)$ becomes as follows;

b'_1	b'_2	b'_3	b'_4	b'_5	b'_6	b'_7	b'_8	b'_9	b'_{10}
(	)	(	(	)	(	(	)	)	)

Fig.**10** String $ass(b)$

Therefore $cont(b)$ and $cont(ass(b))^{(+1)}$ are detected in the following figures,

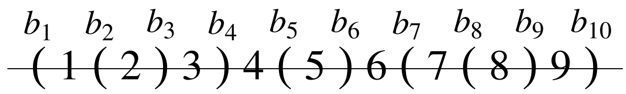

Fig.**11** *cont*(b)

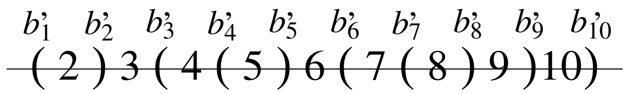

Fig.**12** $cont(ass(b))^{+1}$

and the set of polynomials $f^{(+1)}_{ass(b)}$ is given by

$$f^{(+1)}_{ass(b)} = \{x_2, x_5, x_7 + x_9, x_8, x_4 + x_6 + x_{10}\}$$

Note here there is a strong resemblance between this and the set

$$f_b = \{x_1 + x_3, x_2, x_5, x_7 + x_9, x_8\}.$$

The elements of $f^{(+1)}_{ass(b)}$ except the last one and the elements of f_b except the first one coincide completely. This phenomina is a special case of the following general proposition;

Proposition 6.1. *For any $b \in Par_n$, its associative transformation $ass(b)$ is balanced too, hence it defines a map $ass : Par_n \to Par_n$. Moreover we have*

$$f_b \cap K[x_2, \cdots, x_{2n-1}] = f^{(+1)}_{ass(b)} \cap K[x_2, \cdots, x_{2n-1}],$$

and

$$\begin{aligned}&\#(f_b \setminus (f_b \cap K[x_2, \cdots, x_{2n-1}]))\\ &= \#(f^{(+1)}_{ass(b)} \setminus (f^{(+1)}_{ass(b)} \cap K[x_2, \cdots, x_{2n-1}])) = 1.\end{aligned}$$

Therefore we have

$$\#(f_b \cup f^{(+1)}_{ass(b)}) = n + 1.$$

Proof. This is a direct consequence of the definition of the content and the associative transformation. □

This has a nice consequence for the geometry of the intersection of two Chebyshev varieties:

Theorem 6.1. *For any integer $n \geq 2$ and for any $b \in Par_n$, we have*

$$V(f_b) \cap V(f^{(+1)}_{ass(b)}) \subset V(u[1, 2n-1]) \cap V(u[2, 2n]), \tag{6.1}$$

and the dimension of each side is given by

$$\dim(V(f_b) \cap V(f^{(+1)}_{ass(b)})) = n-1, \tag{6.2}$$
$$\dim(V(u[1,2n-1]) \cap V(u[2,2n])) = 2(n-1). \tag{6.3}$$

Proof. The assertion (6.1) follows from Corrolary 4.1. As for (6.2), recall that $cont(b_i) \cap cont(b_j) = \phi$ for any pair of distinct left brackets $b_i, b_j \in L(b)$ by the very definition of the content. Therefore the linear polynomials f_{b_i} $(b_i \in L(b))$ are linearly independent. Moreover an element of f_b contains the term x_1, and an element of $f^{(+1)}_{ass(b)}$ contains the term x_{2n}, as is noted in Proposition 6.1. Hence the $n+1$ elements of $f_b \cup f^{(+1)}_{ass(b)}$ are linearly independent and the assertion (6.2) follows. The last assertion (6.3) is a direct consequence of the fact that AC-variety is a complete intersection. This completes the proof. □

7 Bra-ketting transformation

In order to construct subvarieties of AC-varieties and to require the dimension of those as large as possible, we introduce another type of transformation of the string of round brackets, called *bra-ketiing transformation.*

Definition 7.1. *For any $b \in Par_n$ with $r(b_1) = b_i$, let $b' = (b'_j)_{1\le j\le 2n+1}$ be the string of bracket defined by*

$$b'_j = \begin{cases} \langle, & j=1, \\ b_j, & 2 \le j \le i-1, \\ |, & j=i, \\ b_j, & i+1 \le j \le 2n, \\ \rangle, & j=2n+1. \end{cases}$$

The string b' is denoted by $bra(b)$, and is called the bra-ketting transformation *of b.*

Example 7.1. For the string $b \in Par_5$ in Example 3.2,

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
(	(	)	)	(	)	(	(	)	)

Fig.13 String $b = (())()(())$

the transformed string $b' = bra(b)$ becomes as follows;

b'_1	b'_2	b'_3	b'_4	b'_5	b'_6	b'_7	b'_8	b'_9	b'_{10}	b'_{11}
$<$	(	)	|	(	)	(	(	)	)	$>$

Fig.14 String $bra(b)$

By the definition, for any $b \in Par_n$, the bra-ketting transformation $bra(b)$ is balanced. Hence it defines a map $bra : Par_n \to Bra_n$. This transformation enables us to construct a linear subvarieties of AC_N when $N \equiv 0 \pmod 4$:

Theorem 7.1. *For any positive integer n and for any $b \in Par_n$, we have*

$$V(f_b) \cap V(f^{(+1)}_{ass(b)}) \subset V(f_{bra(b)}(0)). \tag{7.1}$$

Proof. Let $b' = ass(b)$, $d = bra(b)$. Let $L(b) = \{b_{\ell_j}; 1 \le j \le n\}$ with $\ell_1 < \cdots < \ell_n$ so that $\ell_1 = 1$, and let $r(b_1) = b_i$. It follows from the definition that

$$\begin{aligned} cont(d_1) &= cont(b_1), \\ cont(d_i) &= cont(b'_{i-1})^{(+1)}. \end{aligned}$$

Moreover, for any left bracket b_{ℓ_j} with $j \ge 2$, we have

$$cont(b_{\ell_j}) = cont(b'_{\ell_j - 1})^{(+1)} = cont(d_{\ell_j}).$$

(Note here that $b_i \notin L(b)$.) Therefore we have

$$f_b \cup f^{(+1)}_{b'} = \{f_{b_1}\} \cup \{f^{(+1)}_{b'_{i-1}}\} \cup \{f_{b_{\ell_j}}; 2 \le j \le n\},$$

and

$$\begin{aligned} f_d(0) &= \{f_{b_{\ell_j}}; 2 \le j \le n\} \cup \{f_{d_1}(0)\} \\ &= \{f_{b_{\ell_j}}; 2 \le j \le n\} \cup \{f_{b_1} \cdot f^{(+1)}_{b'_{i-1}}\}. \end{aligned}$$

Hence the inclusion (7.1) holds true. This completes the proof. □

Recall that for any $d \in Bra_n$ and for any $c \in K$, we have $V(f_d(c)) \subset V_{[1,2n]}((-1)^{n-1}(c-1))$ by Theorem 6.1. Hence when n is even, we have

$$V(f_d(0)) \subset V_{[1,2n]}((-1)^n) = V_{[1,2n]}(1). \tag{7.2}$$

Now the AC-variety AC_N is defined to be the intersection of three hypersurfaces $V_{[1,N-1]}$, $V_{[2,N]}$, and $V_{[1,N]}(1)$ in $\mathbb{A}^N$. On the other hand, for any $b \in Par_{2n}$, it follows from Corollary 4.1 that

$$V(f_b) \subset V_{[1,2n-1]}, \tag{7.3}$$

$$V(f^{(+1)}_{ass(b)}) \subset V_{[2,2n]}. \tag{7.4}$$

Hence combining the inclusions (7.1)-(7.4), we arrive at the following:

Theorem 7.2. *For any positive integer $n \equiv 0 \pmod 4$, and for any $b \in Par_{n/2}$, we have*

$$V(f_b) \cap V(f^{(+1)}_{ass(b)}) \subset AC_n.$$

8 Strings with a triple bra-ket

Here a *triple bra-ket* means the string $\langle \mid \mid \rangle$. This will enable us to construct subvarieties of AC_N for an odd integer N. For any integer $n \geq 1$, we define the balancedness of a string $b = (b_1, \cdots, b_{2n+2})$ with $n-1$ pairs of round brackets and with one triple bra-ket as follows. Let $b_i = \langle, b_j = |, b_k = |, b_\ell = \rangle$ with $i < j < k < \ell$. Then b is said to be balanced if the five remaining substrings$(b_1, \cdots, b_{i-1}), (b_{i+1}, \cdots, b_{j-1}), (b_{j+1}, \cdots, b_{k-1}), (b_{k+1}, \cdots, b_{\ell-1})$, and $(b_{\ell+1}, \cdots, b_{2n+2})$ are balanced in the sense of Definition 3.1. Moreover we employ the convention that the left angle bracket b_i matches the first vertical bar b_j, which matches the second vertical bar b_k, which matches the right angle bracket b_ℓ. Accordingly we regard a vertical bar both as a left bracket and as a right bracket, and we can define a partial order on the set $L(b)$ of left brackets in b in the same way as for Par_n. We denote the set of balanced strings of length $2n+2$ with $n-1$ pairs of round brackets and with one triple bra-ket by $Tbra_n$. The content of a string $b \in Tbra_n$ is defined in the same way as for Par_n with the only difference that the content of the triple bra-ket (b_i, b_j, b_k, b_ℓ) is denoted by $\langle cont(b_i) | cont(b_j) | cont(b_k) \rangle$. For $\epsilon \in \{\pm 1\}$, we associate to b_i, b_j, b_k the polynomials $f_{b_i}(\epsilon), f_{b_j}(\epsilon), f_{b_k}(\epsilon)$ defined by

$$\begin{aligned} f_{b_i}(\epsilon) &= \sum_{p \in cont(b_i)} x_p - \epsilon, \\ f_{b_j}(\epsilon) &= \sum_{p \in cont(b_j)} x_p - \epsilon, \\ f_{b_k}(\epsilon) &= \sum_{p \in cont(b_k)} x_p - \epsilon. \end{aligned}$$

For the remaining left brackets $b_m, m \neq i, j, k$, the polynomial f_{b_m} is defined to be the same as in the case of round bracket. Putting them together, we define the set of polynomials $f_b(\epsilon)$ by

$$f_b(\epsilon) = \{f_{b_m}; 1 \leq m \leq n, m \neq i, j, k\} \cup \{f_{b_i}(\epsilon), f_{b_j}(\epsilon), f_{b_k}(\epsilon)\}.$$

The following example should clarify the meaning.

Example 8.1. Let $b \in Tbra_5$ be the following string:

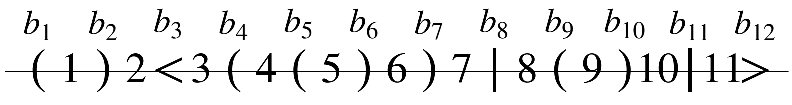

Fig.15 String $b = ()\langle(())|()|\rangle$ with its contents

Then the content and the associated polynomial are given by

$$\begin{aligned} cont(b) &= \{1,5,9,\{4,6\},\langle 3,7|8,10|11\rangle\}, \\ f_b(\epsilon) &= \{x_1,x_5,x_9,x_4+x_6,x_3+x_7-\epsilon,x_8+x_{10}-\epsilon,x_{11}-\epsilon\}. \end{aligned}$$

The family of polynomials $f_b(\epsilon)$ for $b \in Tbra_n$ will play an important role to construct subvarieties of AC-varieties. First we prove the following:

Proposition 8.1. *For any integer $n \geq 1$, $c \in K$, and for any $b \in Tbra_n$, we have*

$$V(f_b(\epsilon)) \subset V_{[1,2n+1]}((-1)^n\epsilon).$$

Proof. We prove this by induction on n. When $n = 1$, the set $Tbra_1$ consists solely of $\langle\ |\ |\ \rangle$. If we call this string b, then its content is given by

$$cont(b) = \{\langle 1|2|3\rangle\}.$$

Therefore we have

$$f_b(\epsilon) = \{x_1-\epsilon, x_2-\epsilon, x_3-\epsilon\}.$$

On the other hand, we have

$$\begin{aligned} V_{[1,3]}((-1)^1\epsilon) &= \{(x_1,x_2,x_3) \in \mathbb{A}^3; u[1,3] = -\epsilon\} \\ &= \{(x_1,x_2,x_3) \in \mathbb{A}^3; x_1x_2x_3 - x_1 - x_3 = -\epsilon\}, \end{aligned}$$

hence the assertion holds since $\epsilon^3 = \epsilon$ for $\epsilon \in \{\pm 1\}$. When $n = 2$, the set $Tbra_2$ has five strings:

$$Tbra_2 = \{()\langle||\rangle, \langle()||\rangle, \langle|()|\rangle, \langle||()\rangle, \langle||\rangle()\}.$$

Let us call these strings b^1, b^2, b^3, b^4, b^5, respectively. Then it follows from the definition that

$$\begin{aligned} f_{b^1}(\epsilon) &= \{x_1, x_3-\epsilon, x_4-\epsilon, x_5-\epsilon\}, \\ f_{b^2}(\epsilon) &= \{x_2, x_1+x_3-\epsilon, x_4-\epsilon, x_5-\epsilon\}, \\ f_{b^3}(\epsilon) &= \{x_3, x_1-\epsilon, x_2+x_4-\epsilon, x_5-\epsilon\}, \\ f_{b^4}(\epsilon) &= \{x_4, x_1-\epsilon, x_2-\epsilon, x_3+x_5-\epsilon\}, \\ f_{b^5}(\epsilon) &= \{x_5, x_1-\epsilon, x_2-\epsilon, x_3-\epsilon\}. \end{aligned}$$

Since $u[1,5] = x_1x_2x_3x_4x_5 - x_1x_2x_3 - x_1x_2x_5 - x_1x_4x_5 - x_3x_4x_5 + x_1 + x_3 + x_5$, we see directly that $V(f_{b^j}(\epsilon)) \subset V_{[1,5]}(\epsilon)$ $(1 \leq j \leq 5)$. Now assume that $n \geq 3$ and that the assertions is proved for any $n' < n$. Take an arbitrary balanced string $b = (b_1, \cdots, b_{2n+2}) \in Tbra_n$. As is done in the proof of Theorem 4.1, we divide our argument into two cases: (I) when every minimal element in $L(b)$ is maximal, and (II) when there exists an element in $L(b)$ which is minimal but not maximal.

Case (I) Every minimal element in $L(b)$ is maximal. In this case the left angle bracket must be odd-indexed one b_{2k-1}, say, and hence we must have $b_{2k} = |$, $b_{2k+1} = |$, and $b_{2k+2} = \rangle$. It follows that

$$\begin{aligned} & cont(b) \\ & = \{\{1\}, \{3\}, \cdots, \{2k-3\}, \langle 2k-1|2k|2k+1\rangle, \{2k+3\}, \cdots, \{2n+1\}\}, \end{aligned}$$

hence we have

$$\begin{aligned} & f_b(\epsilon) \\ & = \{x_1, x_3, \cdots, x_{2k-3}, x_{2k-1} - \epsilon, x_{2k} - \epsilon, x_{2k+1} - \epsilon, x_{2k+3}, \cdots, x_{2n+1}\}. \end{aligned}$$

On the other hand the equalities (1.6) and (1.7) imply that

$$\begin{aligned} & u(0, x_2, 0, \cdots, 0, x_{2k-2}, x_{2k-1}, x_{2k}, x_{2k+1}, x_{2k+2}, 0, x_{2k+4}, \cdots, 0, x_{2n}, 0) \\ = \; & (-1)^{k-1} \cdot (-1)^{n-k} \cdot u(x_{2k-1}, x_{2k}, x_{2k+1}) \\ = \; & (-1)^{n-1}(x_{2k-1}x_{2k}x_{2k+1} - x_{2k-1} - x_{2k+1}). \end{aligned}$$

Therefore if $a = (a_1, \cdots, a_{2n+1}) \in V(f_b(\epsilon))$, then we have

$$u(a_1, \cdots, a_{2n+1}) = (-1)^{n-1} \cdot (-\epsilon) = (-1)^n \epsilon,$$

and hence we have $a \in V_{[1,2n+1]}((-1)^n \epsilon)$.

Case (II): There exists an element in $L(b)$ which is minimal but not maximal. We call such an element b_k. Note that this is not a left angle bracket nor a vertical bar, since these are always maximal by the definition of balancedness. Let b' denote the string defined by

$$b' = (b_1, \cdots, b_{k-1}, b_{k+2}, \cdots, b_{2n+1}).$$

Note that b' belongs to $Tbra_{n-1}$. Let $p : \mathbb{A}^{2n+1} \to \mathbb{A}^{2n-1}$ denote the map defined by

$$p(x_1, \cdots, x_{2n+1}) = (x_1, \cdots, x_{k-2}, x_{k-1} + x_{k+1}, x_{k+2}, \cdots, x_{2n+1}),$$

and let p_k denote its restriction to $V(x_k) \subset \mathbb{A}^{2n+1}$. Then by a similar reasoning to that for (4.2) and (4.2) we see that

$$p_k^{-1}(V(f_{b'}(\epsilon))) \;=\; V(f_b(\epsilon)), \tag{8.1}$$
$$p_k^{-1}(V_{[1,2n-1]}((-1)^{n-1}\epsilon) \;\subset\; V(x_k) \cap V_{[1,2n+1]}((-1)^n \epsilon). \tag{8.2}$$

Since the induction hypothesis implies $V(f_{b'}(\epsilon)) \subset V_{[1,2n-1]}((-1)^{n-1}\epsilon)$, it follows from (8.1) and (8.2) that

$$V(f_b(\epsilon)) \subset V(x_k) \cap V_{[1,2n+1]}((-1)^n \epsilon) \subset V_{[1,2n+1]}((-1)^n \epsilon).$$

This completes the proof for Case (II), hence at the same time the proof of Proposition 8.1. □

9 Associative transformation II

In this section we introduce another kind of associative transformation. This will enable us to construct a family of subvarieties of AC_N for odd N. Let $Bra_{\langle n}$ denote the subset of Bra_n consisting of strings with $b_1 = \langle$. We define *associative transformation for bra-ket* as follows:

Definition 9.1. *For any $b \in Bra_{\langle n}$, suppose that $b_i = |$ and $b_j = \rangle$. Let $b' = (b'_k)_{1 \le k \le 2n+1}$ be the string of bracket defined by*

$$b'_k = \begin{cases} b_{k+1}, & 1 \le k \le i-2, \\ \langle, & k = i-1 \\ b_{k+1}, & i \le k \le j-2, \\ |, & k = j-1, \\ b_{k+1}, & j \le k \le 2n, \\ \rangle, & k = 2n+1. \end{cases}$$

The string b' is denoted by $ass_{\langle | \rangle}(b)$, and is called the associative transformation for bra-ket *of b.*

Example 9.1. For the folloing string $b \in Bra_5$,

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
$<$	(	)	$\vert$	(	(	)	)	$>$	(	)

Fig.**16** String $b = \langle()|(())\rangle()$

the transformed string $b' = ass_{\langle | \rangle}(b)$ becomes as follows;

b'_1	b'_2	b'_3	b'_4	b'_5	b'_6	b'_7	b'_8	b'_9	b'_{10}	b'_{11}
(	)	$<$	(	(	)	)	$\vert$	(	)	$>$

Fig.**17** String $ass_{\langle | \rangle}(b)$

Therefore $f_b(c)$ and $f^{(+1)}_{ass_{\langle | \rangle}(b)}(c)$ are given by

$$f_b(c) = \{x_2, x_6, x_5 + x_7, x_{10}, (x_1 + x_3)(x_4 + x_8) - c\},$$

$$f^{(+1)}_{ass_{\langle | \rangle}(b)}(c) = \{x_2, x_6, x_5 + x_7, x_{10}, (x_4 + x_8)(x_9 + x_{11}) - c\}.$$

Note here too that there is a strong resemblance between $f_b(c)$ and $f^{(+1)}_{ass_{\langle | \rangle}(b)}(c)$, and we have the following proposition which can be proved in the same way as for Proposition 6.1:

Proposition 9.1. *For any $b \in Bra_n$, its associative transformation for bra-ket $ass_{\langle|\rangle}(b)$ is balanced too, hence it defines a map $ass_{\langle|\rangle} : Bra_n \to Bra_n$. Moreover we have*

$$f_b(c) \cap K[x_2, \cdots, x_{2n}] = f^{(+1)}_{ass(b)}(c) \cap K[x_2, \cdots, x_{2n}],$$

and

$$\begin{aligned}&\#(f_b(c) \setminus (f_b(c) \cap K[x_2, \cdots, x_{2n-1}])) \\ &= \#(f^{(+1)}_{ass(b)}(c) \setminus (f^{(+1)}_{ass(b)}(c) \cap K[x_2, \cdots, x_{2n-1}])) \\ &= 1.\end{aligned}$$

Therefore we have

$$\#(f_b(c) \cup f^{(+1)}_{ass(b)}(c)) = n - 1.$$

When we set c to be $\epsilon \in \{\pm 1\}$ in the pair $f_b(c)$ and $f^{(+1)}_{ass_{\langle|\rangle}(b)}(c)$, we find a strong connection between these and the set of polynomial associated to a string $\in Tbra_n$. In order to explain the connection we introduce one more transformation:

Definition 9.2. *For any $b \in Bra_{\langle n}$, suppose that $b_i = |$ and $b_j = \rangle$. Let $b" = (b"_k)_{1 \le k \le 2n+2}$ be the string of bracket defined by*

$$b"_k = \begin{cases} \langle, & k = 1, \\ b_k, & 2 \le k \le i-1, \\ |, & k = i \\ b_k, & i+1 \le k \le j-1, \\ |, & k = j, \\ b_k, & j+1 \le k \le 2n+1, \\ \rangle, & k = 2n+2. \end{cases}$$

The string $b"$ is denoted by $ass_{to\langle||\rangle}(b)$, and is called the associative transformation to triple bra-ket *of b.*

Example 9.2. For the folloing string $b \in Bra_5$ used in Example 9.1,

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
$<$	(	)	|	(	(	)	)	$>$	(	)

Fig.18 String $b = \langle ()|(())\rangle ()$

the transformed string $b" = ass_{to\langle||\rangle}(b)$ becomes as follows;

$b"_1$	$b"_2$	$b"_3$	$b"_4$	$b"_5$	$b"_6$	$b"_7$	$b"_8$	$b"_9$	$b"_{10}$	$b"_{11}$	$b"_{12}$
$<$	(	)	|	(	(	)	)	|	(	)	$>$

Fig.19 String $ass_{to\langle||\rangle}(b)$

The following proposition is crucial for our construction of a family of linear subvarieties of AC_N with N odd:

Proposition 9.2. *For any integer $n \geq 1$ and for any string $b \in Bra_{\langle n}$, let $b' = ass_{\langle|\rangle}(b)$ and $b" = ass_{to\langle||\rangle}(b)$. Then for any $\epsilon \in \{\pm 1\}$, we have*

$$V(f_{b"}(\epsilon)) \subset V(f_b(1)) \cap V(f_{b'}^{(+1)}(1)). \tag{9.1}$$

Proof. Let $b_i = |$ and $b_j = \rangle$. Note that $b_1 = \langle$ by the definition of $Bra_{\langle n}$. Then it follows from the definition that for any $b_k \in L(b) - \{b_1, b_i\}$, we have

$$f_{b_k} = f_{b'_{k-1}}^{(+1)} = f_{b"_k}.$$

Therefore we have

$$\begin{aligned} f_b(1) &= \{f_{b_k}; b_k \in L(b) - \{b_1, b_i\}\} \cup \{f_{b_1}(1)\}, & (9.2)\\ f_{b'}^{(+1)}(1) &= \{f_{b_k}; b_k \in L(b) - \{b_1, b_i\}\} \cup \{f_{b'_{i-1}}^{(+1)}(1)\}, & (9.3)\\ f_{b"}(\epsilon) &= \{f_{b_k}; b_k \in L(b) - \{b_1, b_i\}\} \cup \{f_{b"_1}(\epsilon), f_{b"_i}(\epsilon), f_{b"_j}(\epsilon)\}. & (9.4) \end{aligned}$$

Recall that the polynomials $f_{b_1}(1), \cdots, f_{b"_j}(\epsilon)$ on the right hand sides of the above equalities are defined as

$$\begin{aligned} f_{b_1}(1) &= \sum_{p \in cont(b_1)} x_p \sum_{q \in cont(b_i)} x_q - 1, \\ f_{b'_{i-1}}^{(+1)}(1) &= \sum_{q \in cont(b'_{i-1})} x_{q+1} \sum_{r \in cont(b'_{j-1})} x_{r+1} - 1 \\ &= \sum_{q \in cont(b_i)} x_q \sum_{r \in cont(b'_{j-1})} x_{r+1} - 1, \\ f_{b"_1}(\epsilon) &= \sum_{p \in cont(b"_1)} x_p - \epsilon \\ &= \sum_{p \in cont(b_1)} x_p - \epsilon, \\ f_{b"_i}(\epsilon) &= \sum_{q \in cont(b"_i)} x_q - \epsilon \\ &= \sum_{q \in cont(b_i)} x_q - \epsilon, \\ f_{b"_j}(\epsilon) &= \sum_{r \in cont(b"_j)} x_r - \epsilon \\ &= \sum_{r \in cont(b'_{j-1})} x_{r+1} - \epsilon. \end{aligned}$$

Combining these with (9.2)-(9.4) and noting that $\epsilon^2 = 1$ for $\epsilon \in \{\pm 1\}$, we see that the inclusion (9.1) holds true. This completes the proof. $\square$

This proposition enables us to obtain a family of linear subvarieties of AC_N for an arbitrary odd integer N:

Theorem 9.1. *For any integer $n \geq 1$ and for any string $b \in Bra_{\langle n}$, we have*

$$V(f_{ass_{to\langle||\rangle}(b)}((-1)^n)) \subset AC_{2n+1}. \tag{9.5}$$

Proof. It follows from Corollary 5.1 that

$$V(f_b(1)) \quad \subset \quad V_{[1,2n]}, \tag{9.6}$$
$$V(f^{(+1)}_{ass_{\langle|\rangle}(b)}(1)) \quad \subset \quad V_{[2,2n+1]}. \tag{9.7}$$

Furthermore it follows from Proposition 8.1 that

$$V(f_{ass_{to\langle||\rangle}(b)}((-1)^n)) \subset V_{[1,2n+1]}((-1)^n \cdot (-1)^n) = V_{[1,2n+1]}(1). \tag{9.8}$$

Hence combining (9.6)-(9.8) with Proposition 9.2, we have the inclusion (9.5). This completes the proof. $\square$

10 Strings with a quadruple bra-ket

In order to deal with subvarieties of AC_N with $N \equiv 2 \pmod 4$, we need strings with a *quadruple bra-ket*. Here a quadruple bra-ket means the string $\langle\ |\ |\ |\ \rangle$. For any integer $n \geq 2$, we define the balancedness of a string $b = (b_1, \cdots, b_{2n+1})$ with $n-2$ pairs of round brackets and with one quadruple bra-ket in a way similar to that for strings with a triple bra-ket. Let $b_i = \langle, b_j = |, b_k = |, b_\ell = |, b_m = \rangle$ with $i < j < k < \ell < m$. Then b is said to be balanced if the six remaining substrings$(b_1, \cdots, b_{i-1}), (b_{i+1}, \cdots, b_{j-1}), (b_{j+1}, \cdots, b_{k-1}), (b_{k+1}, \cdots, b_{\ell-1}), (b_{\ell+1}, \cdots, b_{m-1})$, and $(b_{m+1}, \cdots, b_{2n+1})$ are balanced in the sense of (). We regard a vertical bar both as a left bracket and as a right bracket, and we can define a partial order on the set $L(b)$ of left brackets in b in the same way as for Par_n. We denote the set of balanced strings of length $2n+1$ with $n-2$ pairs of round brackets and with one quadruple bra-ket by $Qbra_n$. The content of a string $b \in Qbra_n$ is defined in the same way as for Par_n with the only difference that the content of the bra-ket $(b_i, b_j, b_k, b_\ell, b_m)$ is denoted by $\langle cont(b_i)|cont(b_j)|cont(b_k)|cont(b_\ell)\rangle$. We associate to the vertical bars b_j, b_k, b_ℓ, the polynomials $f_{b_j}(2), f_{b_k}(2), f_{b_\ell}(2)$ defined by

$$\begin{aligned} f_{b_j}(2) &= \sum_{p\in cont(b_i)} x_p \sum_{q\in cont(b_j)} x_q - 2, \\ f_{b_k}(2) &= \sum_{p\in cont(b_j)} x_p \sum_{q\in cont(b_k)} x_q - 2, \\ f_{b_\ell}(2) &= \sum_{p\in cont(b_k)} x_p \sum_{q\in cont(b_\ell)} x_q - 2. \end{aligned}$$

For the remaining left brackets $b_r, r \neq i, j, k, \ell$, the polynomial f_{b_r} is defined to be the same as in the case of round bracket. Putting them together, we define the set of polynomials $f_b(2)$ by

$$f_b(2) = \{f_{b_m}; b_m \in L(b), m \neq i, j, k, \ell\} \cup \{f_{b_j}(2), f_{b_k}(2), f_{b_l}(2)\}.$$

The family of polynomials $f_b(2)$ for $b \in Qbra_n$ will play an important role to construct subvarieties of AC-varieties AC_N with $N \equiv 2 \pmod 4$. First we prove the following:

Proposition 10.1. *For any integer $n \geq 2$ and for any $b \in Qbra_n$, we have*

$$V(f_b(2)) \subset V_{[1,2n]}((-1)^{n-1}).$$

Proof. We prove this by induction on n. When $n = 2$, the set $Qbra_2$ consists solely of $\langle \mid \mid \mid \rangle$. If we call this string b, then its content is given by

$$cont(b) = \{\langle 1|2|3|4\rangle\}.$$

Therefore we have

$$f_b(2) = \{x_1x_2 - 2, x_2x_3 - 2, x_3x_4 - 2\}.$$

Note that if $(a_1, \cdots, a_4) \in V(f_b(2))$, then $a_i \neq 0$ for $i = 1, \cdots, 4$, hence the second and the third equations imply that $a_2 = a_4$ holds. On the other hand, we have

$$\begin{aligned} & V_{[1,4]}((-1)^{2-1}) \\ = & \{(x_1, x_2, x_3, x_4) \in \mathbb{A}^4; u[1, 4] = -1\} \\ = & \{(x_1, x_2, x_3, x_4) \in \mathbb{A}^4; x_1x_2x_3x_4 - x_1x_2 - x_1x_4 - x_3x_4 + 1 = -1\}, \end{aligned}$$

hence if $(a_1, \cdots, a_4) \in V(f_b(2))$, then we have

$$a_1a_2a_3a_4 - a_1a_2 - a_1a_4 - a_3a_4 + 1 = 2 \cdot 2 - 2 - 2 - 2 + 1 = -1,$$

from which the assertion follows. Now assume that $n \geq 3$ and that the assertions is proved for any $n' < n$. Take an arbitrary balanced string $b = (b_1, \cdots, b_{2n+1}) \in Qbra_n$. As is done in the proof of Theorem 4.1, we divide our argument into two cases: (I) when every minimal element in $L(b)$ is maximal, and (II) when there exists an element in $L(b)$ which is minimal but not maximal.

Case (I) Every minimal element in $L(b)$ is maximal. In this case the left angle bracket must be odd-indexed one b_{2k-1}, say, and hence we must have $b_{2k} = |$, $b_{2k+1} = |$, $b_{2k+2} = |$, and $b_{2k+3} = \rangle$. It follows that

$$\begin{aligned} & cont(b) \\ = \quad & \{\{1\},\{3\},\cdots,\{2k-3\},\langle 2k-1|2k|2k+1|2k+2\rangle,\{2k+4\},\cdots,\{2n\}\}, \end{aligned}$$

hence we have

$$\begin{aligned} f_b(2) \quad = \quad & \{x_1, x_3, \cdots, x_{2k-3}, x_{2k-1}x_{2k}-2, x_{2k}x_{2k+1}-2, \\ & \qquad x_{2k+1}x_{2k+2}-2, x_{2k+4}, \cdots, x_{2n}\}. \end{aligned}$$

On the other hand the equalities (1.6) and (1.7) imply that

$$\begin{aligned} & u(0, x_2, 0, \cdots, 0, x_{2k-2}, x_{2k-1}, x_{2k}, x_{2k+1}, x_{2k+2}, x_{2k+3}, \\ & \qquad 0, x_{2k+5}, \cdots, 0, x_{2n-1}, 0) \\ = \quad & (-1)^{k-1} \cdot (-1)^{n-k-1} \cdot u(x_{2k-1}, x_{2k}, x_{2k+1}, x_{2k+2}) \\ = \quad & (-1)^n u(x_{2k-1}, x_{2k}, x_{2k+1}, x_{2k+2}). \end{aligned}$$

Therefore if $a = (a_1, \cdots, a_{2n}) \in V(f_b(2))$, then it follows from our proof for the case $n = 2$ that

$$u(a_1, \cdots, a_{2n+1}) = (-1)^n \cdot (-1) = (-1)^{n-1},$$

and hence we have $a \in V_{[1,2n]}((-1)^{n-1})$.

Case (II): There exists an element in $L(b)$ which is minimal but not maximal. We call such an element b_k. Note that this is not a left angle bracket nor a vertical bar, since those are always maximal by the definition of balancedness. Let b' denote the string defined by

$$b' = (b_1, \cdots, b_{k-1}, b_{k+2}, \cdots, b_{2n+1}).$$

Note that b' belongs to $Qbra_{n-1}$. Let $p : \mathbb{A}^{2n} \to \mathbb{A}^{2n-2}$ denote the map defined by

$$p(x_1, \cdots, x_{2n}) = (x_1, \cdots, x_{k-2}, x_{k-1} + x_{k+1}, x_{k+2}, \cdots, x_{2n}),$$

and let p_k denote its restriction to $V(x_k) \subset \mathbb{A}^{2n}$. Then by a similar reasoning to that for (4.2) and (4.3), we have

$$p_k^{-1}(V(f_{b'}(2))) \quad = \quad V(f_b(2)), \tag{10.1}$$
$$p_k^{-1}(V_{[1,2n-2]}((-1)^{n-2}) \quad \subset \quad V(x_k) \cap V_{[1,2n]}((-1)^{n-1}). \tag{10.2}$$

Since the induction hypothesis implies $V(f_{b'}(2)) \subset V_{[1,2n-2]}((-1)^{n-2})$, it follows from (10.1) and (10.2) that

$$V(f_b(2)) \subset V(x_k) \cap V_{[1,2n]}((-1)^{n-1}) \subset V_{[1,2n]}((-1)^{n-2}).$$

This completes the proof for Case (II), hence at the same time the proof of Proposition 10.1. $\square$

In order to construct a family of subvarieties of AC_N with $N \equiv 2 \pmod 4$, we need to introduce sets of polynomials $f_b(2)$ for $b \in Tbra_n$ too. These are, however, defined in a similar way to those for $b \in Qbra_n$ as follows. For any $b \in Tbra_n$, assume that $b_i = \langle, b_j = |, b_k = |, b_\ell = \rangle$ with $i < j < k < \ell$. We associate to the vertical bars b_j, b_k the polynomials $f_{b_j}(2), f_{b_k}(2)$ defined by

$$\begin{aligned} f_{b_j}(2) &= \sum_{p \in cont(b_i)} x_p \sum_{q \in cont(b_j)} x_q - 2, \\ f_{b_k}(2) &= \sum_{p \in cont(b_j)} x_p \sum_{q \in cont(b_k)} x_q - 2. \end{aligned}$$

For the remaining left brackets $b_m, m \neq i, j, k$, the polynomial f_{b_m} is defined to be the same as in the case of round bracket. Putting them together, we define the set of polynomials $f_b(2)$ by

$$f_b(2) = \{f_{b_m}; b_m \in L(b), m \neq i, j, k\} \cup \{f_{b_j}(2), f_{b_k}(2)\}.$$

Then we can show the following:

Proposition 10.2. *For any integer $n \geq 1$ and for any $b \in Tbra_n$, we have*

$$V(f_b(2)) \subset V_{[1,2n+1]}.$$

Proof. We prove this by induction on n. Since our proof goes similarly to that for Proposition 10.1, we only point out some necessary minor changes to deal with this case. When $n = 1$, the set $Tbra_1$ consists solely of $\langle\,|\,|\,\rangle$. If we call this string b, then its content is given by

$$cont(b) = \{\langle 1|2|3 \rangle\}.$$

Therefore we have

$$f_b(2) = \{x_1x_2 - 2, x_2x_3 - 2\}.$$

Note that if $(a_1, \cdots, a_3) \in V(f_b(2))$, then $a_i \neq 0$ for $i = 1, 2, 3$, hence the first and the second equations imply that $a_1 = a_3$. On the other hand, we have

$$\begin{aligned} V_{[1,3]} &= \{(x_1, x_2, x_3) \in \mathbb{A}^3; u[1,3] = 0\} \\ &= \{(x_1, x_2, x_3) \in \mathbb{A}^3; x_1x_2x_3 - x_1 - x_3 = 0\}, \end{aligned}$$

hence if $(a_1, \cdots, a_3) \in V(f_b(2))$, then we have

$$a_1 a_2 a_3 - a_1 - a_3 = 2a_3 - a_1 - a_3 = 0,$$

from which the assertion holds. The remaining induction process can be proved similarly and we omit it. □

11 Associative transformation III

In this section we introduce one more associative transformation. This will enable us to construct a family of subvarieties of AC_N for $N \equiv 2 \pmod 4$. Let $Tbra_{\langle n}$ denote the subset of $Tbra_n$ consisting of strings with $b_1 = \langle$. We define *associative transformation for triple bra-ket* as follows:

Definition 11.1. *For any $b \in Tbra_{\langle n}$, suppose that $b_i = b_j = |$, and $b_k = \rangle$ with $i < j < k$. Let $b' = (b'_m)_{1 \leq m \leq 2n+1}$ be the string of bracket defined by*

$$b'_m = \begin{cases} b_{m+1}, & 1 \leq m \leq i-2, \\ \langle, & m = i-1 \\ b_{m+1}, & i \leq m \leq j-2, \\ |, & m = j-1, \\ b_{m+1}, & j \leq m \leq k-2, \\ |, & m = k-1, \\ b_{m+1}, & k \leq m \leq 2n, \\ \rangle, & m = 2n+1. \end{cases}$$

The string b' is denoted by $ass_{\langle||\rangle}(b)$, and is called the associative transformation for triple bra-ket *of b.*

As is seen from the definition, this transformation is a natural generalization of that for bra-ket introduced in Definition 9.1. Furthermore we define an associative transformation to quadruple bra-ket of $b \in Tbra_{\langle n}$ as follows:

Definition 11.2. *For any $b \in Tbra_{\langle n}$, suppose that $b_i = |$, $b_j = |$, and $b_k = \rangle$. Let $b" = (b"_m)_{1 \leq m \leq 2n+3}$ be the string of bracket defined by*

$$b"_m = \begin{cases} \langle, & m = 1, \\ b_m, & 2 \leq m \leq i-1, \\ |, & m = i \\ b_m, & i+1 \leq m \leq j-1, \\ |, & m = j, \\ b_m, & j+1 \leq m \leq k-1, \\ |, & m = k, \\ b_m, & k+1 \leq m \leq 2n+2, \\ \rangle, & m = 2n+3. \end{cases}$$

The string $b"$ is denoted by $ass_{to\langle|||\rangle}(b)$, and is called the associative transformation to quadruple bra-ket *of b.*

The following proposition provides us with a crucial ingredient to construct a family of linear subvarieties of AC_N with $N \equiv 2 \pmod 4$:

Proposition 11.1. *For any integer $n \geq 1$ and for any string $b \in Tbra_{\langle n}$, let $b' = ass_{\langle || \rangle}(b)$ and $b" = ass_{to\langle ||| \rangle}(b)$. Then we have*

$$V(f_{b"}(2)) \subset V(f_b(2)) \cap V(f_{b'}^{(+1)}(2)). \tag{11.1}$$

Proof. Let $b_i = |$, $b_j = |$, and $b_k = \rangle$ with $i < j < k$. Then it follows from the definition that for any $b_m \in L(b) - \{b_1, b_i, b_j\}$, we have

$$f_{b_m} = f_{b'_{m-1}}^{(+1)} = f_{b"_m}.$$

Therefore we have

$$\begin{aligned} f_b(2) &= \{f_{b_m}; b_m \in L(b) - \{b_1, b_i, b_j\}\} \cup \{f_{b_i}(2), f_{b_j}(2)\}, && (11.2)\\ f_{b'}^{(+1)}(2) &= \{f_{b_m}; b_m \in L(b) - \{b_1, b_i, b_j\}\} \cup \{f_{b'_{j-1}}^{(+1)}(2), f_{b'_{k-1}}^{(+1)}(2)\}, && (11.3)\\ f_{b"}(2) &= \{f_{b_m}; b_m \in L(b) - \{b_1, b_i, b_j\}\}\\ &\quad \cup\{f_{b"_i}(2), f_{b"_j}(2), f_{b"_k}(2)\}. && (11.4) \end{aligned}$$

Recall that the polynomials $f_{b_i}(2), \cdots, f_{b"_k}(2)$ on the right hand sides of the above equalities are defined as

$$\begin{aligned} f_{b_i}(2) &= \sum_{p \in cont(b_1)} x_p \sum_{q \in cont(b_i)} x_q - 2,\\ f_{b_j}(2) &= \sum_{q \in cont(b_i)} x_q \sum_{r \in cont(b_j)} x_r - 2,\\ f_{b'_{j-1}}^{(+1)}(2) &= \sum_{q \in cont(b'_{i-1})} x_{q+1} \sum_{r \in cont(b'_{j-1})} x_{r+1} - 2\\ &= \sum_{q \in cont(b_i)} x_q \sum_{r \in cont(b_j)} x_r - 2,\\ f_{b'_{k-1}}^{(+1)}(2) &= \sum_{r \in cont(b'_{j-1})} x_{r+1} \sum_{s \in cont(b'_{k-1})} x_{s+1} - 2\\ &= \sum_{r \in cont(b_j)} x_r \sum_{s \in cont(b'_{k-1})} x_{s+1} - 2,\\ f_{b"_i}(2) &= \sum_{p \in cont(b_1)} x_p \sum_{q \in cont(b_i)} x_q - 2,\\ f_{b"_j}(2) &= \sum_{q \in cont(b_i)} x_q \sum_{r \in cont(b_j)} x_r - 2,\\ f_{b"_k}(2) &= \sum_{r \in cont(b_j)} x_r \sum_{s \in cont(b'_{k-1})} x_{s+1} - 2. \end{aligned}$$

Combining these with (11.2)-(11.4), we see that the inclusion (11.1) holds true. This completes the proof. □

This proposition enables us to obtain a family of linear subvarieties of AC_N for an arbitrary integer $N \equiv 2 \pmod 4$:

Theorem 11.1. *For any integer $n \geq 1$ and for any string $b \in Tbra_{\langle n}$, we have*

$$V(f_{ass_{to\langle|||\rangle}(b)}(2)) \subset V_{[1,2n+1]} \cap V_{[2,2n+2]} \cap V_{[1,2n+2]}((-1)^n). \tag{11.5}$$

In particular if n is an even integer $2m$, say, then we have

$$V(f_{ass_{to\langle|||\rangle}(b)}(2)) \subset AC_{4m+2}. \tag{11.6}$$

Proof. It follows from Proposition 10.2 that

$$V(f_b(2)) \quad \subset \quad V_{[1,2n+1]}, \tag{11.7}$$

$$V(f^{(+1)}_{ass_{\langle|\rangle}(b)}(2)) \quad \subset \quad V_{[2,2n+2]}. \tag{11.8}$$

Furthermore it follows from Proposition 10.1 that

$$V(f_{ass_{to\langle|||\rangle}(b)}(2)) \subset V_{[1,2n+2]}((-1)^n). \tag{11.9}$$

Hence combining (11.6)-(11.8) with Proposition 11.1, we have the inclusion (11.5). This completes the proof. □

12 Examples of AC-polygons

Having obtained special subvarieties of AC_N for any $N \geq 3$ in the previous sections, we are interested in the shapes of AC-polygons which corresponds to these subvarieties. In this section we describe some examples of n-gons with area center for small n when the base field $K = \mathbb{R}$.

12.1 Triangle

As is seen in Example 1.1, every triangle has an area center, which coincides with its barycenter.

12.2 Quadrangle

As is seen in Example 1.2, the AC-variety AC_4 consists of two lines $\ell_1 = V(x_1, x_3, x_2+x_4)$ and $\ell_2 = V(x_2, x_4, x_1+x_3)$ in $\mathbb{A}^4$. On the other hand it follows from Theorem 7.2 that for any $b \in Par_2$ we have $V(f_b) \cap V(f^{(+1)}_{ass(b)}) \subset AC_4$.

Furthermore note that Par_2 consists of two strings $b^1 = ()()$ and $b^2 = (())$. For the string b^1, we have

$$\begin{aligned} f_{b^1} &= \{x_1, x_3\}, \\ f^{(+1)}_{ass(b^1)} &= \{x_3, x_2 + x_4\}, \end{aligned}$$

and for the string b^2, we have

$$\begin{aligned} f_{b^2} &= \{x_2, x_1 + x_3\}, \\ f^{(+1)}_{ass(b^2)} &= \{x_2, x_4\}. \end{aligned}$$

Therefore we see that

$$\begin{aligned} V(f_{b^1}) \cap V(f^{(+1)}_{ass(b^1)} &= \ell_1, \\ V(f_{b^2}) \cap V(f^{(+1)}_{ass(b^2)} &= \ell_2. \end{aligned}$$

Thus our bracket subvarieties exactly provide us with the two irreducible components of AC_4. Noting that ℓ_1 is parametrized as $\{(0, t, 0, -t); t \in \mathbb{R}\}$, we denote by $P^1(p_0, p_1; t)$ the quadrangle $p_0p_1p_2p_3$ for arbitrary p_0 and p_1 which corresponds to $(a_1, a_2, a_3, a_4) = (0, t, 0, -t)$ through (1.2). Then we see that $P^1(p_0, p_1; t)$ is specified by the equations

$$\begin{aligned} p_2 &= -p_0, \\ p_3 &= -tp_0 - p_1. \end{aligned}$$

Similarly we denote by $P^2(p_0, p_1; t)$ the quadrangle which corresponds to the variable point $(t, 0, -t, 0) \in \ell_2$. Then it is specified by the equations

$$\begin{aligned} p_2 &= -p_0 + tp_1, \\ p_3 &= -p_1. \end{aligned}$$

We illustrate a few examples of $P^1(p_0, p_1; t)$ and $P^2(p_0, p_1; t)$ with $p_0 = (1, 0)$ and $p_1 = (1, 1)$ below:

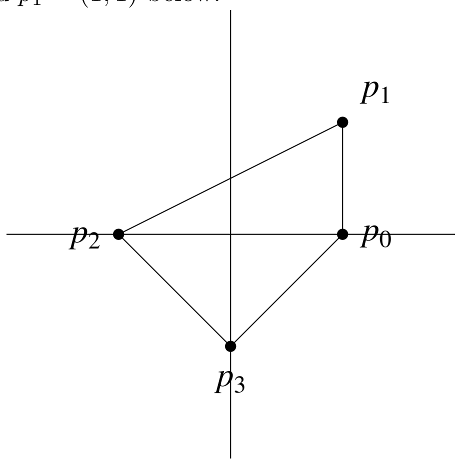

Fig.20 $P^1((1,0),(1,1);-1)$

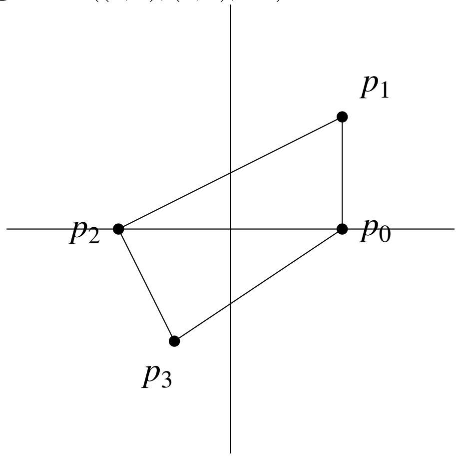

Fig.21 $P^1((1,0),(1,1);-1/2)$

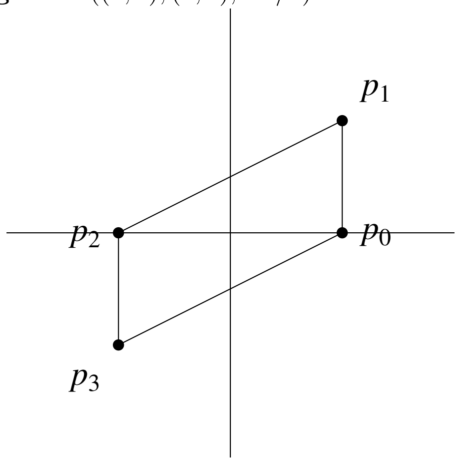

Fig.22 $P^1((1,0),(1,1);0)$

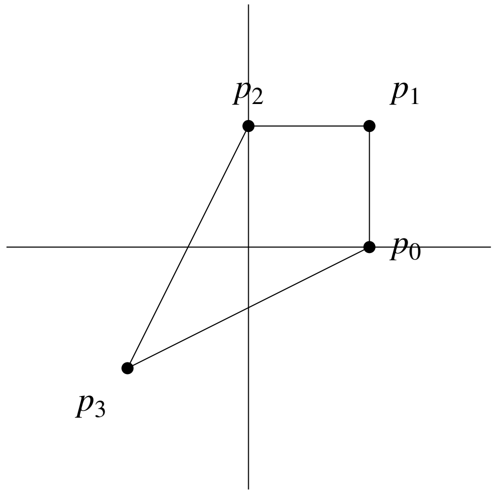

Fig.23 $P^2((1,0),(1,1);1)$

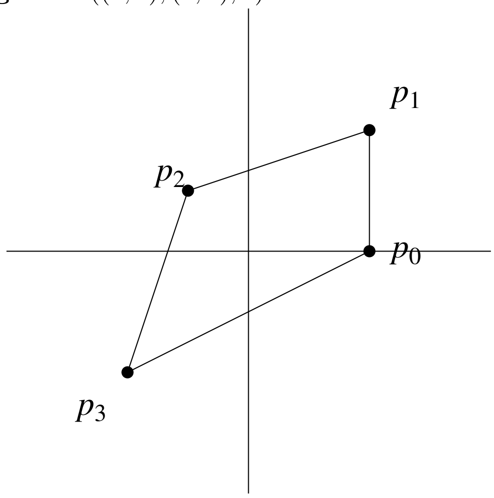

Fig.24 $P^2((1,0),(1,1);1/2)$

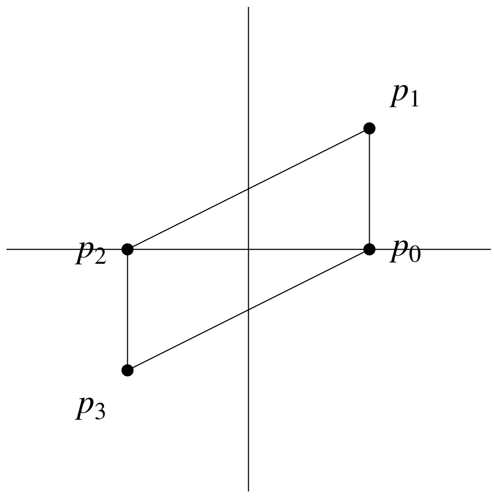

Fig.25 $P^2((1,0),(1,1);0)$

As is seen in Fig.* and Fig.**, the intersection $\ell_1 \cap \ell_2 = \{(0,0,0,0)\}$ corresponds to a parallelogram for any initial points p_0, p_1.

12.3 Pentagon

We refer the reader to Theorem 9.1 with $n = 2$. According to this, we see that for any $b \in Bra_{\langle 2}$ the associative transformation $ass_{to\langle||\rangle}(b)$ provides us with AC-pentagon. For example, if we take $b^1 = \langle()|\rangle \in Bra_{\langle 2}$, then we have $ass_{to\langle||\rangle}(b^1) = \langle()||\rangle$ and

$$f_{ass_{to\langle||\rangle}(b^1)}(1) = \{x_2, x_1 + x_3 - 1, x_4 - 1, x_5 - 1\}.$$

Hence it defines a line

$$V(f_{ass_{to\langle||\rangle}(b^1)}(1)) = \{(t, 0, 1-t, 1, 1) \in \mathbb{A}^5; t \in \mathbb{R}\}$$

in $\mathbb{A}^5$. Let $P^3(p_0, p_1; t)$ denote the pentagon which corresponds to the point $(t, 0, 1-t, 1, 1)$ on this line. The following two figures show the corresponding pentagons with $t = 1, \dfrac{1}{2}$. Here we set $p_0 = (1,0), p_1 = (1,1)$ too:

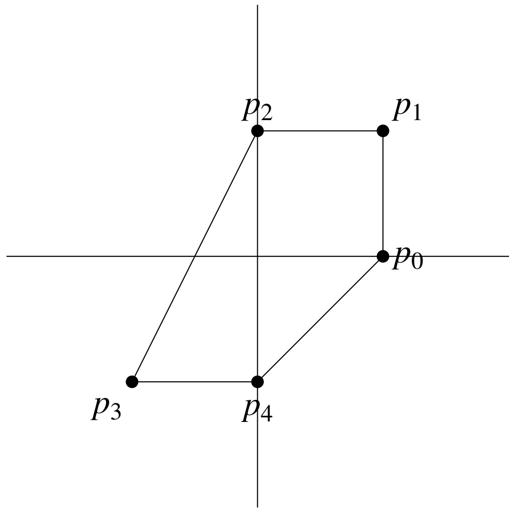

Fig.**26** $P^3((1,0),(1,1);1)$

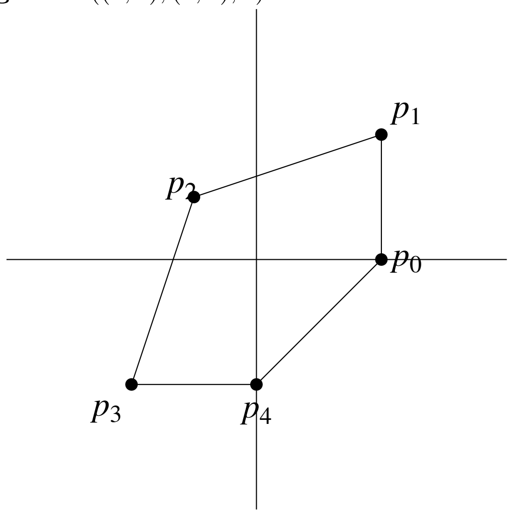

Fig.**27** $P^3((1,0),(1,1);1/2)$

Observe that the latter has a symmetry with an axis through the point p_2. The next example sets $b^2 = \langle|()\rangle \in Bra_{\langle 2}$, and hence $ass_{to\langle||\rangle}(b) = \langle|()|\rangle$ with

$$f_{ass_{to\langle||\rangle}(b^2)}(1) = \{x_3, x_1 - 1, x_2 + x_4 - 1, x_5 - 1\}.$$

Hence it defines a line

$$V(f_{ass_{to\langle||\rangle}(b^2)}(1)) = \{(1, t, 0, 1 - t, 1,) \in \mathbb{A}^5; t \in \mathbb{R}\}$$

in $\mathbb{A}^5$. Let $P^4(p_0, p_1; t)$ denote the pentagon which corresponds to the point $(1, t, 0, 1 - t, 1)$ on this line. The following figure shows the corresponding pentagons with $t = 1$. Here we set $p_0 = (1, 0), p_1 = (1, 1)$ too:

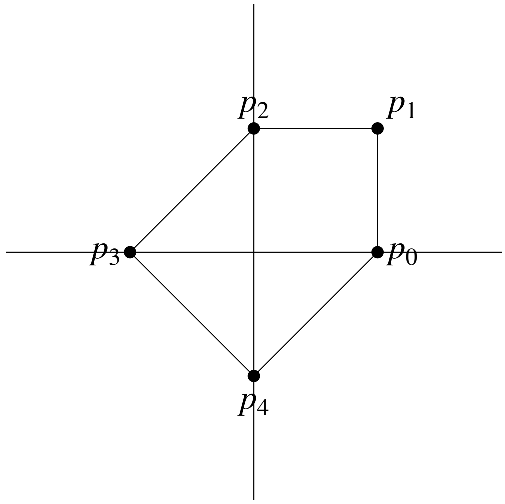

Fig.28 $P^4((1,0),(1,1);1)$

12.4 Hexagon

We refer the reader to Theorem11.1, (11.6) with $m=1$. Among several options, we choose $b = \langle |()| \rangle \in Tbra_{\langle 2}$. Then we have $ass_{to\langle |||\rangle}(b) = \langle |()||\rangle$, and

$$f_{ass_{to\langle |||\rangle}(b)}(2) = \{x_3, x_1(x_2+x_4)-2, (x_2+x_4)x_5-2, x_5x_6-2\}.$$

As another set of generators of the ideal generated by $f_{ass_{to\langle |||\rangle}(b)}(2)$, we can take

$$\{x_3, x_1-x_5, x_2+x_4-x_6, x_5x_6-2\},$$

and hence it defines a quadric surface in $\mathbb{A}^6$. Therefore we can parametrize $V(f_{ass_{to\langle |||\rangle}(b)}(2))$ as

$$V(f_{ass_{to\langle |||\rangle}(b)}(2)) = \{(s,t,0,\frac{2}{s}-t,s,\frac{2}{s}); s,t \in \mathbb{R}\}.$$

Let $P^5(p_0,p_1;s,t)$ denote the hexagon with initial points p_0, p_1, which corresponds to the point $(s,t,0,\frac{2}{s}-t,s,\frac{2}{s})$ on this surface. The following figure shows the corresponding hexagon with $s=\frac{4}{3}, t=\frac{3}{2}$. Here we set $p_0 = (1,0), p_1 = (1,1)$ too:

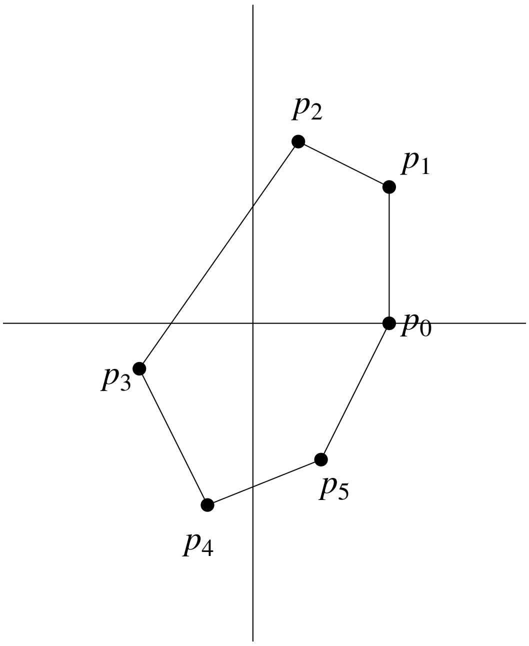

Fig.29 $P^5((1,0),(1,1);\frac{4}{3},\frac{3}{2})$

12.5 Heptagon

We refer the reader to Theorem 9.1 with $n = 3$. According to this, we see that for any $b \in Bra_{\langle 3}$ the associative transformation $ass_{to\langle||\rangle}(b)$ provides us with AC-heptagon. For example, if we take $b = \langle()|\rangle() \in Bra_{\langle 3}$, then we have $ass_{to\langle||\rangle}(b) = \langle()||()\rangle$ and

$$f_{ass_{to\langle||\rangle}(b)}(-1) = \{x_2, x_6, x_1 + x_3 + 1, x_4 + 1, x_5 + x_7 + 1\}.$$

Hence it defines a plane

$$V(f_{ass_{to\langle||\rangle}(b)}(-1)) = \{(s, 0, -s-1, -1, t, 0, -t-1) \in \mathbb{A}^7; s, t \in \mathbb{R}\}$$

in $\mathbb{A}^7$. Let $P^6(p_0, p_1; s, t)$ denote the heptagon which corresponds to the point $(s, 0, -s-1, -1, t, 0, -t-1)$ on this plane. The following figure shows the corresponding heptagon with $(s,t) = (-\frac{1}{2}, -\frac{1}{2}$. Here we set $p_0 = (1,0), p_1 = (1,0)$:

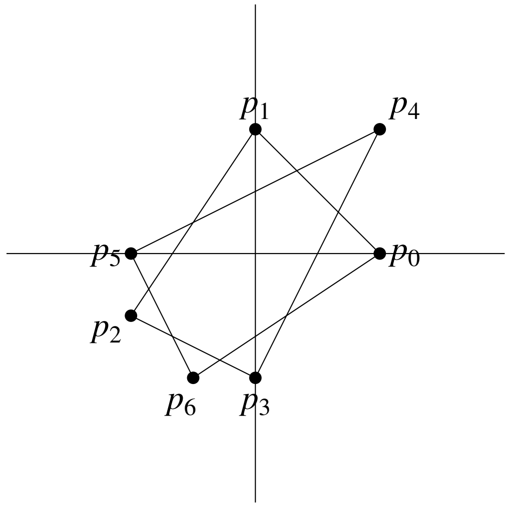

Fig.30 $P^6((1,0),(0,1);-\frac{1}{2},-\frac{1}{2})$

Remark. Note that the string $ass_{to\langle||\rangle}(b) = \langle()||()\rangle$ has bilateral symmetry. Furthermore the values of parameters $(s,t) = (-\dfrac{1}{2},-\dfrac{1}{2})$ provides us with the point $(x_1,\cdots,x_7) = (-\dfrac{1}{2},0,-\dfrac{1}{2},-1,-\dfrac{1}{2},0,-\dfrac{1}{2}) \in V(f_{ass_{to\langle||\rangle}(b)}(-1))$, which has bilateral symmetry too. These facts contribute to the symmetry of the corresponding heptagon.

Remark. The winding number of this heptagon is equal to two. It might be an interesting problem to find the winding numbers of the AC-polygons which correspond to the points on our subvarieties. We hope to return to this problem in a forthcoming paper.

12.6 Regular polygon

The regular n-gons (including stars) are obviously examples of AC-polygons. These are obtained from our viewpoint as follows. Let $\Delta = \{(x,\cdots,x) \in \mathbb{A}^n; x \in \mathbb{R}\}$, the diagonal of $\mathbb{A}^n$. Then the intersection $V(u[1,n]) \cap \Delta$ gives rise to the regular n-gons:

Proposition 12.1. *For any $n \geq 3$ and for any $k \in [0,n-1]$, let $c_{(n;k)} = \cos\frac{2k}{n}\pi$. Then we have*

$$AC_n \cap \Delta = \{(c_{(n;k)},\cdots,c_{(n;k)}); k \in [0,n-1]\}.$$

Proof. As is explained in [, Proposition 2.7], we have

$$u[1,n]\Big|_{x_1=x,\cdots,x_n=x} = U_n(\frac{x}{2}), \tag{12.1}$$

where $U_n(z)$ denotes the Chebyshev polynomial of the second kind defined by

$$U_n(z) = \frac{\sin(n+1)\theta}{\sin\theta} \quad \textit{with} \;\; z = \cos\theta. \tag{12.2}$$

Since AC_n is defined to be $V(u[1,n]-1, u[1,n-1], u[2,n])$, it follows from (12.1) that

$$AC_n \cap \Delta = V(U_{n-1}\left(\frac{x}{2}\right), U_n\left(\frac{x}{2}-1\right)\}.$$

Let us put $x = 2\cos\theta$, then it follows from (12.2) that

$$\begin{aligned} U_{n-1}\left(\frac{x}{2}\right) &= \frac{\sin n\theta}{\sin\theta}, \\ U_n\left(\frac{x}{2}\right) &= \frac{\sin(n+1)\theta}{\sin\theta}. \end{aligned}$$

Hence by a simple computation we see that the equality in Proposition 12.1 holds true. □

Now for any integer m, we have by the addition formulas

$$\begin{aligned} & 2\cos\theta(\cos m\theta, \sin m\theta) - (\cos(m-1)\theta, \sin(m-1)\theta) \\ = \; & (\cos(m+1)\theta, \sin(m+1)\theta). \end{aligned}$$

Hence if we take the initial points p_0, p_1 as

$$\begin{aligned} p_0 &= (1,0), \\ p_1 &= (\cos\frac{2k}{n}\pi, \sin\frac{2k}{n}\pi), \end{aligned}$$

and set $x_1 = \cdots = x_n = c_{(n;k)} = 2\cos\frac{2k}{n}\pi$, then we have

$$p_m = (\cos\frac{2km}{n}\pi, \sin\frac{2km}{n}\pi), m \in \mathbb{Z}.$$

Thus the n-gon $P = (p_0, \cdots, p_{n-1})$ corresponding to $(c_{(n;k)}, \cdots, c_{(n;k)})$ gives rise to the regular n-star $\{n/k\}$ in Schläfli symbol.

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