



MATHEMATICAL ANALYSIS OF MELODIES: SLOPE AND DISCRETE FRÉCHET DISTANCE

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Abstract

A directed graph, called an M-graph, is attached to every melody. Our chief concern in this paper is to investigate: (1) how the positivity of the slope of the M-graph is related to singability of the melody, (2) when the M-graph has a symmetry, and (3) how we can detect a similarity between two melodies. For the third theme, we introduce the notion of transposed discrete Fréchet distance, and show its relevance in the study of similarity detection among an arbitrary set of melodies.

0. Introduction

In the article [2], the authors introduced a method of attaching a graph to an arbitrary melody. We call it here the *M-graph* of the melody. By using the M-graph of melody as a main ingredient, we investigate in this paper: (1) how the positivity of the slope of the M-graph is related to *singability* of the melody, (2) when the M-graph has a symmetry, and (3) how we can detect a similarity between two melodies. Accordingly, we divide the paper into three parts. In the first part, we focus on the slope of the M-graph, which is defined

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by the method of least squares, and investigate how the slope is related to musical characteristics of the original melody. For example, among melodies which are composed of six notes C4, D4, E4, F4, G4, A4 and begin with C4, the largest slope is attained by (C4, D4, E4, F4, G4, A4) with slope 0.986 and the smallest slope is attained by (C4, A4, D4, G4, E4, F4) with slope -0.729 . One can see that the latter is harder to sing than the former. Through the analysis of several data including this example, we will show that the positivity of the slope is strongly related to its singability. One word of caution: We do not assert that positivity of the slope is related to its goodness. For example, the first phrase of the most famous nocturne (in E \flat major) by Chopin has a (slightly) negative slope -0.089 , but cannot be claimed that it is a bad melody accordingly. We see, however, that all of the fifteen other nocturnes by Chopin have positive slopes (see Table 8). We also consider how the slope of the M-graph is changed under transposition, inversion, and retrograde of the original melody. In the second part of the paper, we investigate how a symmetry of the M-graph is reflected to the character of the melody. We invite the reader to have a look at Figure 2, which is the M-graph of the basic twelve-tone row of the string quartet Op. 28 by Webern. This amazing example leads us to the main theorem (Theorem 2.1) of the second part, which characterizes the melodies with symmetric M-graph in terms of a certain arithmetic property. In the third part of the paper, we propose a distance, called *transposed discrete Fréchet distance*, and show its relevance for similarity detection through several examples. We invite the reader again to catch a glimpse of Figure 8, the data of which comes from the author's questionnaire to the students in a class on discrete geometry. The national anthem of Israel, "Twinkle, twinkle, little star", and the Japanese classical song "Kojo no Tsuki", which constitute the nearest cluster, can be sung simultaneously and quite harmoniously. This surprise motivated him to write this paper.

1. Slope of M-graph

1.1. Definition of M-graph

In order to express a melody by a definite sequence of integers, we let C4

(middle C) correspond to 0, C#4 to 1, and so on. In this way, we can associate a sequence of integers with each melody. For example the melody “C4, D4, F4, E4”, which is the main theme of the fourth movement of the Jupiter symphony by Mozart, corresponds to the sequence “0, 2, 5, 4”. From now on we identify a melody of finite length with the sequence of integers of finite length which is constructed by this rule. Furthermore, to any sequence $\mathbf{a} = (a_1, a_2, \dots, a_n)$ of integers, we attach a sequence of points $\mathbf{p} = (p_1, p_2, \dots, p_{n-1})$ with $p_i \in \mathbf{R}^2$ ($1 \leq i \leq n-1$) by the following rule:

$$p_1 = (a_1, a_2), p_2 = (a_2, a_3), \dots, p_{n-1} = (a_{n-1}, a_n).$$

Let $G(\mathbf{a}) = (V(\mathbf{a}), E(\mathbf{a}))$ be the directed graph with the set of vertices

$$V(\mathbf{a}) = (p_1, p_2, \dots, p_{n-1}),$$

and the set of edges

$$E(\mathbf{a}) = \{(p_1, p_2), (p_2, p_3), \dots, (p_{n-2}, p_{n-1})\}.$$

We call $G(\mathbf{a})$ the *M-graph* associated to the melody \mathbf{a} . (“M” stands for melody.) When $\mathbf{a} = (0, 2, 5, 4)$, for example, its M-graph $G(\mathbf{a})$ is depicted as follows:

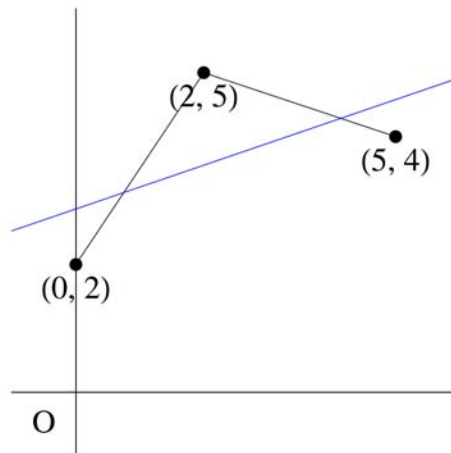


Figure 1. M-graph of Jupiter.

The line which cuts through the M-graph in this figure is obtained by the

least squares fitting. Its slope will be referred as *the slope* of the melody, and denoted by $s(\mathbf{a})$. In this case we see that $s(\mathbf{a}) = 0.342$.

Remark. A formula for the slope in the method of least squares will be recalled in Proposition 1.1.

1.2. Distribution of slopes of M-graphs

We will show that there exists a correlation between the slope of a melody and its *singability*. Let us look at the set

$$M_4 =$$

$$\{(0, 2, 4, 5), (0, 2, 5, 4), (0, 4, 2, 5), (0, 4, 5, 2), (0, 5, 2, 4), (0, 5, 4, 2)\},$$

which collects all the melodies consisting of C4, D4, E4, F4 beginning with C4. The slopes of these are computed as follows:

Table 1. Slopes of four-tone melodies

melody	note names	slope
(0,2,4,5)	(C,D,E,F)	0.750
(0,2,5,4)	(C,D,F,E)	0.342
(0,4,2,5)	(C,E,D,F)	-0.500
(0,4,5,2)	(C,E,F,D)	-0.214
(0,5,2,4)	(C,F,D,E)	-0.605
(0,5,4,2)	(C,F,E,D)	-0.357

Notice here that our friend (C, D, F, E) has the second highest slope among the melodies in M_4 , and that the other melodies, except the simplest melody (C, D, E, F), have negative slopes. In order to understand what is going on, we take next the set M_5 of melodies consisting of C4, D4, E4, F4, G4 beginning with C4. The top three melodies with largest slope and the bottom three with smallest slope are tabulated below:

Table 2. Largest three slopes

ranking	melody	note names	slope
1st	(0,2,4,5,7)	(C,D,E,F,G)	0.915
2nd	(0,2,5,4,7)	(C,D,F,E,G)	0.576
3rd	(0,2,4,7,5)	(C,D,E,G,F)	0.467

Table 3. Smallest three slopes

ranking	melody	note names	slope
-1st	(0,7,2,5,4)	(C,G,D,F,E)	-0.655
-2nd	(0,7,2,4,5)	(C,G,D,E,F)	-0.617
-3rd	(0,7,4,5,2)	(C,G,E,F,D)	-0.538

The next table shows the top three and the worst three of slopes among melodies which consists of six notes C4, D4, E4, F4, G4, A4 and begins with C4:

Table 4. Largest three slopes

ranking	melody	note names	slope
1st	(0,2,4,5,7,9)	(C,D,E,F,G,A)	0.98630
2nd	(0,2,5,4,7,9)	(C,D,F,E,G,A)	0.81507
3rd	(0,2,4,7,5,9)	(C,D,E,G,F,A)	0.64384
3rd	(0,4,2,5,7,9)	(C,E,D,F,G,A)	0.64384

Table 5. Smallest three slopes

ranking	melody	note names	slope
-1st	(0,9,2,7,4,5)	(C,A,D,G,E,F)	-0.72932
-2nd	(0,7,4,5,2,9)	(C,G,E,F,D,A)	-0.72603
-3rd	(0,9,2,7,5,4)	(C,A,D,G,F,E)	-0.69925

As the reader may notice in these examples, melodies with large (positive) slope tend to be easy to sing and those with small (negative) slope are hard to sing. The table below describes the numbers of melodies with positive, negative, or zero slope in each category:

Table 6. Distribution of slopes

constituent	positive	negative	zero
{C,D,E,F,G}	8	16	0
{C,D,E,F,G,A}	45	75	0
{C,D,E,F,G,A,B}	262	457	1

We notice that, in each category, the number of melodies with negative slope is about twice the number of those with positive slope. Therefore we may assert that composers choose instinctively melodies with positive slope, which constitute rather a minor part in the world of melodies, in order to make their works singable ones.

Keeping these observations in mind, we examine the slopes of actual melodies composed by two great composers, Schumann and Chopin. Table 7 shows the slopes of the first phrases of the sixteen songs in “Dichterliebe” by Schumann:

Table 7. Distribution of slopes in the song cycle Dichterliebe

No.	1	2	3	4	5	6	7	8
slope	0.183	0.302	0.951	0.553	0.545	0.712	0.438	0.584

No.	9	10	11	12	13	14	15	16
slope	0.691	0.543	0.656	0.316	0.929	-0.572	0.450	0.666

Among these songs, only one song has a negative slope. It is the fourteenth song, titled “Allnächtlich im Traume seh’ ich dich,” whose slope is -0.572 . The fact that, when we listen to the song cycle as a whole, we feel a certain soothing effect at this 14th song, might be related to the negativity of its slope.

The following table shows the slopes of all the nocturnes composed by Chopin:

Table 8. Distribution of slopes in nocturnes

No.	1	2	3	4	5	6	7	8
slope	0.980	-0.089	0.371	0.508	0.860	0.667	0.496	0.677

No.	9	10	11	12	13	14	15	16
slope	0.419	0.641	0.673	0.650	0.970	0.520	0.293	0.099

Among these nocturnes, only the second one has a negative slope. This nocturne also has a kind of soothing effect, which might be one of the reasons why this is widely regarded as the most popular Nocturne by Chopin. On the other hand the largest and the second largest slopes are attained by the first one (in B \flat minor) and 13-th one (in C minor), respectively. Both pieces move us (or at least the author) with their distinctive deep sorrow.

Next we examine the motif $M_J = (C4, D4, \underline{F4}, \underline{E4}, A4, A4, A4, G4)$ of the Jupiter from a different viewpoint. We fix the beginning note C4 and the ending note G4, and compute the slopes of every possible permutation of the remaining notes. As a result we find that the highest slope is attained by the melody $M_{J'} = (C4, D4, \underline{E4}, \underline{F4}, A4, A4, A4, G4)$ with its slope 0.657, and the second highest by the original M_J with its slope 0.572. The change of the order of E4 and F4 in $M_{J'}$ seems to give a nuance and to make M_J more attractive.

By a similar computation using the motif $M_{39} = (G5, Ab5, Bb5, Ab5, G5, F5, \underline{Eb5}, \underline{F5}, Bb4)$ of the fourth movement of the 39th symphony by Mozart, we find that the highest slope is attained by $M_{39'} = (G5, Ab5, Bb5, Ab5, G5, F5, \underline{F5}, \underline{Eb5}, Bb4)$ with its slope 1.461. Here again the change of the underlined pair in $M_{39'}$ composes the original M_{39} , whose slope is equal to 1.048.

How about the Pastoral Symphony by Beethoven? The motif $(A4, Bb4, \underline{D5}, \underline{C5}, Bb4, A4, G4, C4)$ of the first movement turns out to have the highest slope 1.366 among the permutations with fixed the first and the last notes. In fact an equal record is achieved by the melody $(A4, Bb4, \underline{C5}, \underline{D5}, Bb4, A4, G4, C4)$. The latter is attractive too and may be used as an ingredient of a good song.

1.3. Slopes under transformations

In this subsection, we consider what occurs to the slope of a melody if it is transposed, inverted, or reversed.

For an arbitrary melody $\mathbf{x} = (x_1, \dots, x_{n+1})$ and for any $t \in \mathbb{Z}$, let $\mathbf{x} + t = (x_1 + t, \dots, x_{n+1} + t)$, the transposition by t . Furthermore we denote the inversion $(-x_1, \dots, -x_{n+1})$ by \mathbf{x}^i , and the retrograde (x_{n+1}, \dots, x_1) by \mathbf{x}^r . Here we recall the formula for the slope of a point data based on the method of least squares:

Proposition 1.1. Let P denote a set of points $(x_1, y_1), \dots, (x_N, y_N)$ on \mathbb{R}^2 . Then the slope $s(P)$ obtained through the method of least squares is given by the formula

$$s(P) = \frac{N \sum_{i=1}^N x_i y_i - \sum_{i=1}^N x_i \sum_{i=1}^N y_i}{N \sum_{i=1}^N x_i^2 - \left(\sum_{i=1}^N x_i \right)^2}. \quad (1.1)$$

(I) Transposition. The M-graph of transposition $M(\mathbf{x} + t)$ consists of the points $(x_i + t, x_{i+1} + t)$ ($1 \leq i \leq n$). Hence we have $M(\mathbf{x} + t) = M(\mathbf{x}) + (t, t)$, namely all of the points in $M(\mathbf{x} + t)$ are translations of the ones in $M(\mathbf{x})$ by one and the same point (t, t) . Therefore by the very definition of the method of least squares we have the following:

Proposition 1.2. For any melody \mathbf{x} , we have

$$s(M(\mathbf{x} + t)) = s(M(\mathbf{x})).$$

Remark. One can prove this by a direct computation of the slope on the right hand side by using the formula (1.1).

(II) Inversion. Since the numerator and the denominator of the right hand side of (1.1) are homogeneous polynomials of degree two in the variables x_i, y_i ($1 \leq i \leq N$), both of them are invariant under the transformation $x_i \mapsto -x_i, y_i \mapsto -y_i$ ($1 \leq i \leq N$). Hence we have the following:

Proposition 1.3. For any melody \mathbf{x} , we have

$$s(M(\mathbf{x}^i)) = s(M(\mathbf{x})).$$

Combination of Propositions 1.2 and 1.3 yields the following:

Corollary 1.1. For any melody \mathbf{x} and for any $t \in \mathbb{Z}$, we have

$$s(M(\mathbf{x}^i + t)) = s(M(\mathbf{x})).$$

For example, if $\mathbf{x} = (\text{A4}, \text{C5}, \text{B4}, \text{A4}, \text{E5}) = (9, 12, 11, 9, 16)$ (Paganini),

then $\mathbf{x}^i + 17 = (8, 5, 6, 8, 1) = (\text{Ab4}, \text{F4}, \text{Gb4}, \text{Ab4}, \text{Db4})$ (Rachmaninov). It follows from Corollary 1.1 that their slopes coincide. Actually, one can see that $s(M(\mathbf{x})) = -1.333 < 0$, and passingly that $s(M(\text{Bb3}, \text{C4}, \text{Db4}, \text{Ab3})) = -1.071 < 0$, but that their concatenation satisfies $s(M(\text{Ab4}, \text{F4}, \text{Gb4}, \text{Ab4}, \text{Db4}, \text{Bb3}, \text{C4}, \text{Db4}, \text{Ab3})) = 0.668$. Thus, Rachmaninov composed this fascinating melody with positive slope by combining the two parts with negative slope.

(III) Retrograde. It turns out to be essential to deal with the numerator and the denominator of the slope separately. Accordingly, we set for any melody $\mathbf{x} = (x_1, \dots, x_{n+1})$,

$$N(\mathbf{x}) = n \sum_{i=1}^n x_i x_{i+1} - \sum_{i=1}^n x_i \sum_{i=2}^{n+1} x_i,$$

$$D(\mathbf{x}) = n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i \right)^2,$$

which are obtained by setting $y_i = x_{i+1}$ ($i = 1, \dots, n$) and $N = n$ in (1.1). Let us put $\mathbf{x}^r = (x'_1, \dots, x'_{n+1})$ so that $x'_i = x_{n+2-i}$ ($1 \leq i \leq n+1$). First we look at the numerator $N(\mathbf{x})$:

Proposition 1.4. *For any melody \mathbf{x} , we have*

$$N(\mathbf{x}^r) = N(\mathbf{x}).$$

Proof. This can be proved by the following straightforward computation:

$$\begin{aligned} N(\mathbf{x}^r) &= n \sum_{i=1}^n x'_i x'_{i+1} - \sum_{i=1}^n x'_i \sum_{i=2}^{n+1} x'_i \\ &= n \sum_{i=1}^n x_{n+2-i} x_{(n+2)-(i+1)} - \sum_{i=1}^n x_{n+2-i} \sum_{i=2}^{n+1} x_{n+2-i} \end{aligned}$$

(by letting $i' = n + 1 - i$)

$$\begin{aligned}
&= n \sum_{i'=1}^n x_{i'+1} x_{i'} - \sum_{i'=1}^n x_{i'+1} \sum_{i'=0}^{n-1} x_{i'+1} \\
&= N(\mathbf{x}). \quad \square
\end{aligned}$$

The denominator is, however, not invariant under the retrograde transformation:

Proposition 1.5. *For any melody \mathbf{x} , we have*

$$D(\mathbf{x}^r) = D(\mathbf{x})$$

if and only if

$$x_1 = x_{n+1} \text{ or } (n+1)(x_{n+1} + x_1) = 2 \sum_{i=1}^{n+1} x_i.$$

Proof. We compute the difference $D(\mathbf{x}^r) - D(\mathbf{x})$:

$$\begin{aligned}
D(\mathbf{x}^r) - D(\mathbf{x}) &= \left(n \sum_{i=1}^n (x'_i)^2 - \left(\sum_{i=1}^n x'_i \right)^2 \right) \\
&\quad - \left(n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i \right)^2 \right) \\
&= \left(n \sum_{i=2}^{n+1} x_i^2 - \left(\sum_{i=2}^{n+1} x_i \right)^2 \right) \\
&\quad - \left(n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i \right)^2 \right) \\
&= n(x_{n+1}^2 - x_1^2) + \left(\left(\sum_{i=1}^n x_i \right)^2 - \left(\sum_{i=2}^{n+1} x_i \right)^2 \right) \\
&= n(x_{n+1} - x_1)(x_{n+1} + x_1)
\end{aligned}$$

$$\begin{aligned}
& + \left(\sum_{i=1}^n x_i - \sum_{i=2}^{n+1} x_i \right) \left(\sum_{i=1}^n x_i + \sum_{i=2}^{n+1} x_i \right) \\
& = n(x_{n+1} - x_1)(x_{n+1} + x_1) \\
& \quad + (x_1 - x_{n+1}) \left(2 \sum_{i=1}^{n+1} x_i - x_1 - x_{n+1} \right) \\
& = (x_{n+1} - x_1) \left((n+1)(x_{n+1} + x_1) - 2 \sum_{i=1}^{n+1} x_i \right).
\end{aligned}$$

Hence the assertion follows. \square

Combining Propositions 1.4 and 1.5, we have the following:

Corollary 1.2. *When a melody \mathbf{x} begins and ends with one and the same note, \mathbf{x} and its retrograde have the same slope.*

1.4. Locality of the slope function

In this subsection, we will see that the slope of a melody is determined by the slopes of its parts. More precisely, we show the following:

Proposition 1.6. *For any melody $\mathbf{x} = (x_1, \dots, x_{n+1})$ and for any k with $1 \leq k \leq n-1$, let \mathbf{x}^k denote the triple (x_k, x_{k+1}, x_{k+2}) and let $s_k = s(M(\mathbf{x}^k))$. Then the slope $s(M(\mathbf{x}))$ of the whole melody is a rational function in s_1, \dots, s_{n-1} .*

Proof. It follows from Corollary 1.1 that $s(M(\mathbf{x})) = s(M(\mathbf{x} - x_1))$. Hence if we put $y_i = x_{i+1} - x_i$ ($1 \leq i \leq n$), then we have

$$\begin{aligned}
s(M(\mathbf{x})) & = s \left(M \left(0, y_1, y_1 + y_2, \dots, \sum_{i=1}^n y_i \right) \right) \\
& = \frac{N \left(0, y_1, y_1 + y_2, \dots, \sum_{i=1}^n y_i \right)}{D \left(0, y_1, y_1 + y_2, \dots, \sum_{i=1}^n y_i \right)},
\end{aligned}$$

the rightmost side being the ratio of homogeneous quadratic polynomials in y_1, \dots, y_n . Hence dividing the numerator and the denominator by y_1^2 , we see that $s(M(\mathbf{x}))$ is a quotient of quadratic polynomials in $y_2/y_1, y_3/y_1, \dots, y_n/y_1$. Hence it is a rational function in $y_2/y_1, y_3/y_2, \dots, y_n/y_{n-1}$. On the other hand, we see that

$$\begin{aligned} s_k &= s(M(\mathbf{x}^k)) = s(\{(x_k, x_{k+1}), (x_{k+1}, x_{k+2})\}) \\ &= \frac{x_{k+2} - x_{k+1}}{x_{k+1} - x_k} \\ &= \frac{y_{k+1}}{y_k} \end{aligned}$$

holds for any k with $1 \leq k \leq n-1$. This completes the proof. \square

Example 1.1. When $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is a melody of length 4, we have

$$\begin{aligned} s(M(\mathbf{x})) &= \frac{2(x_1x_2 + x_2x_3 + x_3x_4) - (x_1 + x_2 + x_3)(x_2 + x_3 + x_4)}{3(x_1^2 + x_2^2 + x_3^2) - (x_1 + x_2 + x_3)^2} \\ &= \frac{y_2^2 + 2y_1y_2 + y_1y_3 + 2y_2y_3}{2(y_1^2 + y_1y_2 + y_2^2)} \\ &= \frac{(y_2/y_1)^2 + 2(y_2/y_1) + (y_3/y_1) + 2(y_2/y_1)(y_3/y_1)}{2(1 + (y_2/y_1) + (y_2/y_1)^2)} \\ &= \frac{s_1^2 + 2s_1 + s_1s_2 + 2s_1^2s_2}{2(1 + s_1 + s_1^2)}. \end{aligned}$$

Remark. For an arbitrary finite set of points P in the plane, the slope $s(P)$ is not necessarily a function in the slopes of consecutive segments. For example, let $P = \{(0, 0), (1, 0), (2, 1)\}$. Then the slopes of consecutive segments are 0 and 1, and the whole slope is computed to be

$$s(P) = \frac{3 \cdot 2 - 3 \cdot 1}{3 \cdot 5 - 3^2} = \frac{1}{2}.$$

On the other hand if we put $P' = \{(0, 0), (1, 0), (3, 2)\}$, then the consecutive slopes are 0 and 1, and hence the local slopes coincide with those of P . The whole slope, however, turns out to be

$$s(P') = \frac{3 \cdot 6 - 4 \cdot 2}{3 \cdot 10 - 4^2} = \frac{5}{7}.$$

Thus the slope $s(P)$ is not generally a function of local slopes.

2. Symmetry of M-graphs

In this section, we investigate for what kind of melodies their associated M-graphs have reflective symmetries.

For a line ℓ in the plane \mathbb{R}^2 , let $ref_\ell : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote the reflection with the line ℓ as a set of fixed points. We introduce the following:

Definition 2.1. For any melody $\mathbf{x} = (x_1, \dots, x_{n+1})$, let

$$M(\mathbf{x}) = (p_1, \dots, p_n)$$

be its M-graph so that $p_i = (x_i, x_{i+1})$ for $i = 1, \dots, n$. The melody \mathbf{x} is said to have a *reflective symmetry* if there exists a line ℓ such that $ref_\ell(p_i) = p_{n+1-i}$ holds for any $i \in [1, n]$.

For example, when $\mathbf{x} = (0, 1, 2, \dots, n)$, then one can see that \mathbf{x} has a reflective symmetry with respect to the line $y = -x + n$. We want to characterize the set of melodies with reflective symmetry. For this purpose we need a transformation formula. When ℓ is defined by the equation $y = ax + b$, we have

$$ref_\ell(x, y) = \left(\frac{(1 - a^2)x + 2ay - 2ab}{1 + a^2}, \frac{2ax - (1 - a^2)y + 2b}{1 + a^2} \right). \quad (2.1)$$

On the other hand, when ℓ is parallel to the y-axis and hence defined by the equation $x = c$, we have

$$ref_\ell(x, y) = (-x + 2c, y).$$

In the present paper, we restrict our attention to the melodies without repetition, namely those with pairwise distinct entries. First we deal with the melodies of even length $2n$, and we denote a general melody by indexing it as

$$\mathbf{x} = (x_{-n}, x_{-(n-1)}, \dots, x_{-1}, x_1, \dots, x_{n-1}, x_n).$$

This will ease our description of an inductive argument. We start with the case $n = 2$.

Proposition 2.1. *A melody $\mathbf{x} = (x_{-2}, x_{-1}, x_1, x_2)$ without repetition has a reflective symmetry if and only if the following condition is satisfied:*

$$(I) \quad x_2 = -x_{-2} + x_{-1} + x_1, \quad (2.2)$$

or

$$(II) \quad x_2 = x_{-2} - x_{-1} + x_1. \quad (2.3)$$

The respective axis of symmetry is given by

$$(I) \quad y = -x + x_{-1} + x_1, \quad (2.4)$$

$$(II) \quad y = \frac{x_{-2} - x_1}{x_{-2} - 2x_{-1} + x_1} x - \frac{(x_{-1} - x_1)(x_{-2} + x_1)}{x_{-2} - 2x_{-1} + x_1}. \quad (2.5)$$

Proof. Let ℓ be the axis of symmetry. Then the following two conditions must be met:

$$ref_{\ell}(x_{-2}, x_{-1}) = (x_1, x_2), \quad (2.6)$$

$$ref_{\ell}(x_{-1}, x_1) = (x_{-1}, x_1). \quad (2.7)$$

By our assumption, we have $x_{-1} \neq x_2$, and hence the condition (2.6) implies that ℓ is not parallel to the y -axis. Let $y = ax + b$ be its defining equation. It follows from the formula (2.1) that the condition (2.6) leads us to the following simultaneous equation:

$$\begin{cases} \frac{(1 - a^2)x_{-2} + 2ax_{-1} - 2ab}{1 + a^2} = x_1, \\ \frac{2ax_{-2} - (1 - a^2)x_{-1} + 2b}{1 + a^2} = x_2. \end{cases} \quad (2.8)$$

By multiplying $1 + a^2$ on both sides of these equations, we have

$$\begin{cases} (x_{-2} + x_1)a^2 - 2x_{-1}a + 2ab - x_{-2} + x_1 = 0, \\ (x_{-1} - x_2)a^2 + 2x_{-2}a + 2b - x_{-1} - x_2 = 0. \end{cases} \quad (2.9)$$

By subtracting the first equation from the second equation multiplied by a , we obtain

$$(x_{-1} - x_2)a^3 + (x_{-2} - x_1)a^2 + (x_{-1} - x_2)a + (x_{-2} - x_1) = 0,$$

namely we have

$$(a^2 + 1)((x_{-1} - x_2)a + (x_{-2} - x_1)) = 0.$$

Since a is a real number and $x_{-1} - x_2 \neq 0$, we see that

$$a = \frac{x_{-2} - x_1}{-x_{-1} + x_2}. \quad (2.10)$$

Inserting this expression into the second equation of (2.9), we find that

$$b = \frac{x_{-2}^2 + x_{-1}^2 - x_1^2 - x_2^2}{2(x_{-1} - x_2)}. \quad (2.11)$$

Furthermore, the condition (2.7) with these values for a and b is expressed as the equalities

$$\begin{aligned} & x_{-2}^3 - x_{-1}x_{-2}^2 - x_1x_{-2}^2 + x_{-1}^2x_{-2} - x_1^2x_{-2} - x_2^2x_{-2} + 2x_1x_2x_{-2} \\ & + x_{-1}^3 + x_1^3 + x_{-1}x_1^2 + x_{-1}x_2^2 + x_1x_2^2 - x_{-1}^2x_1 - 2x_{-1}^2x_2 - 2x_1^2x_2 \\ = & (x_{-2}^2 - 2x_1x_{-2} + x_{-1}^2 + x_1^2 + x_2^2 - 2x_{-1}x_2)x_{-1}, \\ & x_{-1}^3 - 2x_{-2}x_{-1}^2 + x_1x_{-1}^2 - x_2x_{-1}^2 + x_{-2}^2x_{-1} - x_1^2x_{-1} - x_2^2x_{-1} \\ & + 2x_{-2}x_2x_{-1} + x_1^3 + x_2^3 - 2x_{-2}x_1^2 - x_1x_2^2 + x_{-2}^2x_1 - x_{-2}^2x_2 + x_1^2x_2 \\ = & (x_{-2}^2 - 2x_1x_{-2} + x_{-1}^2 + x_1^2 + x_2^2 - 2x_{-1}x_2)x_1. \end{aligned}$$

These equations are factored, somewhat miraculously, as

$$\begin{aligned}(x_{-2} - x_1)(x_{-2} - x_{-1} - x_1 + x_2)(x_{-2} - x_{-1} + x_1 - x_2) &= 0, \\ (x_{-1} - x_2)(x_{-2} - x_{-1} - x_1 + x_2)(x_{-2} - x_{-1} + x_1 - x_2) &= 0.\end{aligned}$$

Since $x_{-1} - x_2 \neq 0$, the second equation implies that $x_{-2} - x_{-1} - x_1 + x_2 = 0$ or $x_{-2} - x_{-1} + x_1 - x_2 = 0$, and both alternatives satisfy the first equation. Hence we have

$$(I) \ x_2 = -x_{-2} + x_{-1} + x_1, \text{ or } (II) \ x_2 = x_{-2} - x_{-1} + x_1. \quad (2.12)$$

In case of (I), the slope a and the y-intercept b are found through (2.10) and (2.11) to be

$$\begin{aligned}a &= \frac{x_{-2} - x_1}{-x_{-1} + (-x_{-2} + x_{-1} + x_1)} = \frac{x_{-2} - x_1}{-x_{-2} + x_1} = -1, \\ b &= \frac{x_{-2}^2 + x_{-1}^2 - x_1^2 - (-x_{-2} + x_{-1} + x_1)^2}{2(x_{-1} - (-x_{-2} + x_{-1} + x_1))} \\ &= \frac{-2x_1^2 + 2(x_{-2}x_{-1} + x_{-2}x_1 - x_{-1}x_1)}{2(x_{-2} - x_1)} \\ &= \frac{2(x_{-2} - x_1)(x_{-1} + x_1)}{2(x_{-2} - x_1)} \\ &= x_{-1} + x_1.\end{aligned}$$

This shows that the axis of symmetry in this case is given by (2.4), and the reflection map is given by

$$ref_\ell : (x, y) \mapsto (-y + x_{-1} + x_1, -x + x_{-1} + x_1).$$

Therefore, the condition (2.2) is also sufficient for the reflective symmetry of \mathbf{x} . In case of (II), a similar computation based on (2.10) and (2.11) shows that (2.5) holds true. Furthermore, we notice the following interesting phenomenon in this case: The triangle $p_1p_2p_3$ is an isosceles right triangle with $\angle p_1p_2p_3 = 90^\circ$. For we have

$$p_2 - p_1 = (x_{-1} - x_{-2}, x_1 - x_{-1}),$$

$$p_3 - p_2 = (x_1 - x_{-1}, x_2 - x_1) = (x_1 - x_{-1}, x_{-2} - x_{-1}),$$

which are transversal and have equal lengths. Therefore, the melody $\mathbf{x} = (x_{-2}, x_{-1}, x_1, x_{-2} - x_{-1} + x_1)$ has a reflective symmetry with the bisector of $\angle p_1 p_2 p_3$ as the axis of symmetry. It follows that the condition (2.3) is also sufficient for the melody $\mathbf{x} = (x_{-2}, x_{-1}, x_1, x_2)$ to have a reflective symmetry. This completes the proof. \square

The following corollary can be deduced easily from Proposition 2.1, but it will facilitate our inductive argument later:

Corollary 2.1. *If a melody $\mathbf{x} = (x_{-2}, x_{-1}, x_1, x_2)$ has a reflective symmetry, then we have*

$$x_2 - x_1 = x_{-1} - x_{-2} \text{ or } x_2 - x_1 = x_{-2} - x_{-1}.$$

Next we consider the melodies with six notes.

Proposition 2.2. *A melody $\mathbf{x} = (x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3)$ without repetition has a reflective symmetry if and only if*

$$x_{-i} + x_i \text{ is constant for } i = 1, 2, 3.$$

The axis of symmetry is given by

$$y = -x + x_{-1} + x_1,$$

and the reflection map is given by

$$(x, y) \mapsto (-y + x_{-1} + x_1, -x + x_{-1} + x_1).$$

Proof. Since the submelody $(x_{-2}, x_{-1}, x_1, x_2)$ also has a reflective symmetry, we are in the case (I) or (II) in Proposition 2.1.

(I) The case when $x_2 = -x_{-2} + x_{-1} + x_1$: the reflection in this case is given by

$$ref_\ell(x, y) = (-y + x_{-1} + x_1, -x + x_{-1} + x_1),$$

and hence we must have

$$\text{ref}_\ell(x_{-3}, x_{-2}) = (-x_{-2} + x_{-1} + x_1, -x_{-3} + x_{-1} + x_1) = (x_2, x_3).$$

Therefore we have $x_{-3} + x_3 = x_{-1} + x_1$.

(II) The case when $x_2 = x_{-2} - x_{-1} + x_1$: Let q_{24} (resp. q_{15}) denote the midpoint of p_2p_4 (resp. p_1p_5). Recalling that

$$p_1 = (x_{-3}, x_{-2}), \quad p_2 = (x_{-2}, x_{-1}),$$

$$p_4 = (x_1, x_{-2} - x_{-1} + x_1), \quad p_5 = (x_{-2} - x_{-1} + x_1, x_3),$$

we have

$$q_{24} = \left(\frac{x_{-2} + x_1}{2}, \frac{x_{-2} + x_1}{2} \right),$$

$$q_{15} = \left(\frac{x_{-3} + x_{-2} - x_{-1} + x_1}{2}, \frac{x_{-2} + x_3}{2} \right).$$

Hence we have

$$\overrightarrow{p_3q_{24}} = \left(\frac{x_{-2} - 2x_{-1} + x_1}{2}, \frac{x_{-2} - x_1}{2} \right), \quad (2.13)$$

$$\overrightarrow{q_{24}q_{15}} = \left(\frac{x_{-3} - x_{-1}}{2}, \frac{-x_1 + x_3}{2} \right). \quad (2.14)$$

Note that these vectors are nonzero by our assumption. Furthermore, since these two vectors have the same direction with the axis of symmetry, the equality

$$\overrightarrow{p_3q_{24}} = k \overrightarrow{q_{24}q_{15}} \quad (2.15)$$

holds for some $k \in \mathbb{R}^*$. Since the submelody $(x_{-3}, x_{-2}, x_2, x_3)$ must have the same axis of symmetry as the one for $(x_{-2}, x_{-1}, x_1, x_2)$, it follows from Corollary 2.1 that we necessarily have

$$x_3 - x_2 = x_{-3} - x_{-2}, \quad (2.16)$$

or

$$x_3 - x_2 = x_{-2} - x_{-3}. \quad (2.17)$$

Accordingly, we divide our argument further into two cases:

Case (II.A). The case when $x_3 - x_2 = x_{-3} - x_{-2}$. Since we are in the case when $x_2 = x_{-2} - x_{-1} + x_1$, we have

$$x_3 - (x_{-2} - x_{-1} + x_1) = x_{-3} - x_{-2},$$

which implies that

$$x_3 - x_1 - x_{-3} - x_{-1}.$$

Hence we have

$$\overrightarrow{q_{24}q_{15}} = \left(\frac{x_{-3} - x_{-1}}{2}, \frac{-x_1 + x_3}{2} \right) = \frac{x_{-3} - x_{-1}}{2} (1, 1).$$

This implies by (2.15) and (2.14) that the x -coordinate and the y -coordinate of $\overrightarrow{p_3q_{24}}$ must coincide, and hence it follows from (2.13) that

$$x_{-2} - 2x_{-1} + x_1 = x_{-2} - x_1.$$

This implies that $x_{-1} = x_1$, which contradicts to our assumption.

Case (II.B). The case when $x_3 - x_2 = x_{-2} - x_{-3}$. Note that the vector $\overrightarrow{p_1p_5}$ and the vector $\overrightarrow{p_3q_{24}}$ are transversal by symmetry assumption. Since in our case

$$\overrightarrow{p_1p_5} = (x_2, x_3) - (x_{-3}, x_{-2}) = (x_2 - x_{-3}) \cdot (1, 1) \neq (0, 0),$$

it follows from (2.13) that the inner product of $\overrightarrow{p_1p_5}$ and $\overrightarrow{p_3q_{24}}$, which is equal to

$$\begin{aligned} & \frac{x_2 - x_3}{2} ((x_2 - 2x_{-1} + x_1) + (x_2 - x_1)) \\ &= (x_2 - x_3)(x_2 - x_{-1}), \end{aligned}$$

must become zero. This, however, contradicts to our assumption. Hence both of Case (II.A) and Case (II.B) cannot occur, and the proof is completed. \square

Now we can generalize Proposition 2.2 to an arbitrary melody of even length:

Theorem 2.1. *For any integer $n \geq 3$, a melody $\mathbf{x} = (x_{-n}, \dots, x_{-1}, x_1, \dots, x_n)$ without repetition of length $2n$ has a reflective symmetry if and only if it satisfies the following condition:*

$$(I)_n : x_{-i} + x_i \text{ is constant for } i = 1, \dots, n.$$

When this condition is met, the axis ℓ of symmetry is the line defined by $y = -x + x_{-1} + x_1$ and the reflection map is given by

$$\text{ref}_\ell : (x, y) \mapsto (-y + x_{-1} + x_1, -x + x_{-1} + x_1).$$

Proof. We prove this by induction on n . When $n = 3$, this is Proposition 2.2 itself. When $n \geq 4$, suppose that a melody $\mathbf{x} = (x_{-n}, \dots, x_{-1}, x_1, \dots, x_n)$ has a reflective symmetry. Then the submelody $(x_{-(n-1)}, \dots, x_{-1}, x_1, \dots, x_{n-1})$ also has a reflective symmetry. Then by the induction hypothesis, the assertion $(I)_{n-1}$ holds true. Then we have

$$x_{-(n-1)} + x_{n-1} = x_{-1} + x_1. \quad (2.18)$$

Since the reflection in this case is given by

$$\text{ref}_\ell(x, y) = (-y + x_{-1} + x_1, -x + x_{-1} + x_1),$$

we must have

$$\begin{aligned} \text{ref}_\ell(x_{-n}, x_{-(n-1)}) &= (-x_{-(n-1)} + x_{-1} + x_1, -x_{-n} + x_{-1} + x_1) \\ &= (x_{n-1}, x_n). \end{aligned}$$

The equality of the first entries is assured by (2.18), and that for the second entries is equivalent to

$$x_{-n} + x_n = x_{-1} + x_1. \quad (2.19)$$

Hence the assertion $(I)_n$ holds. Conversely, suppose that the condition $(I)_n$ holds, and let ℓ be the line defined by $y = -x + x_{-1} + x_1$. Then the reflection map with the axis of symmetry ℓ is given by

$$\text{ref}_\ell : (x, y) \mapsto (-y + x_{-1} + x_1, -x + x_{-1} + x_1).$$

It follows that, for any $k \in [1, n]$, we have

$$\begin{aligned} \text{ref}_\ell(x_{-k}, x_{-(k-1)}) &= (-x_{-(k-1)} + x_{-1} + x_1, -x_{-k} + x_{-1} + x_1) \\ &= (x_{k-1}, x_k), \end{aligned}$$

which shows that the melody \mathbf{x} has a reflective symmetry with the axis of symmetry ℓ . This completes the proof. \square

Remark. For a melody of odd length without repetition, we can show the following result: When $n \geq 2$, a melody $\mathbf{x} = (x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_n)$ without repetition has a reflective symmetry if and only if $x_{-i} + x_i = 2x_0$ for any $i \in [1, n]$. This can be proved in a similar way to that for Theorem 2.1, so we omit the proof.

Example 2.1. Webern based his string quartet Op. 28 on the following row:

$$M_W = (7, 6, 9, 8, 12, 13, 10, 11, 15, 14, 17, 16).$$

Amazingly, this melody of length twelve turns out to satisfy the condition $(I)_6$ in Theorem 2.1. Hence it must have a reflective symmetry. We illustrate below the M-graph of its transposition

$$M_{W'} = (1, 0, 3, 2, 6, 7, 4, 5, 9, 8, 11, 10).$$

(Note that transposing does not change the reflective property of the original melody.)

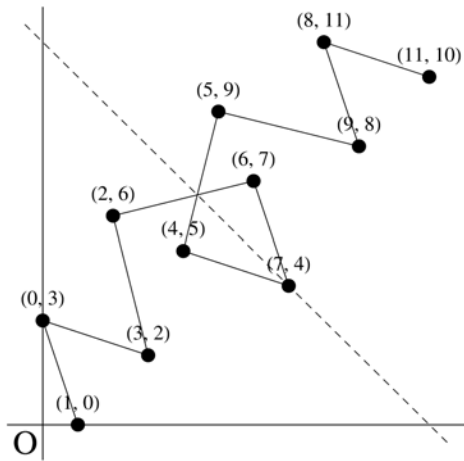


Figure 2. String quartet Op. 28 by Webern.

The dashed line is the axis of symmetry of $M_{W'}$ and is defined by the equation $y = -x + 11$.

Example 2.2. The following row is used in Ode to Napoleon Op. 41 by Shönberg:

$$M_S = (1, 0, 4, 5, 9, 8, 3, 2, 6, 7, 11, 10).$$

Here again we are surprised that this melody satisfies the condition $(I)_6$ in Theorem 2.1. Hence it has a reflective symmetry:

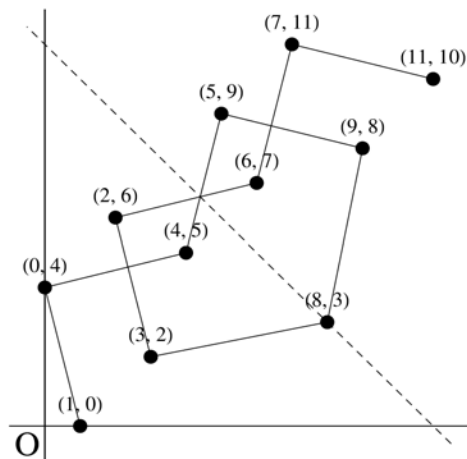


Figure 3. Ode to Napoleon Op. 41 by Shönberg.

The dashed line is the axis of symmetry of M_S and is defined by the equation $y = -x + 11$.

It is needless to say that every twelve-tone row does not have a reflective symmetry. Thus these two composers arrived at the above symmetrical rows through their musical intellect and instinct.

3. Transposed Discrete Fréchet Distance

In this section, we introduce the notion of *transposed discrete Fréchet distance*, abbreviated as TDFD. This is based on the discrete Fréchet distance, abbreviated as DFD. We will show the relevance of TDFD for similarity detection among a given set of melodies.

3.1. Definition of DFD and TDFD

First we recall the definition of DFD for the convenience of the reader. (See [1, 3] for details.) Let $P = (p_1, p_2, \dots, p_n)$ and $Q = (q_1, q_2, \dots, q_m)$ be a pair of sequences of points in \mathbb{R}^2 . A *coupling* L between P and Q is a sequence

$$(p_{a_1}, q_{b_1}), (p_{a_2}, q_{b_2}), \dots, (p_{a_k}, q_{b_k})$$

of distinct pairs from $P \times Q$ such that $a_1 = b_1 = 1$, $a_k = n$, $b_k = m$, and for any $i = 1, \dots, k - 1$, we have $a_{i+1} = a_i$ or $a_{i+1} = a_i + 1$, and $b_{i+1} = b_i$ or $b_{i+1} = b_i + 1$. The *length* $\|L\|$ of the coupling L is defined by

$$\|L\| = \max_{i=1, \dots, k} d(p_{a_i}, q_{b_i}),$$

where $d(*, *)$ denotes the Euclid distance on \mathbf{R}^2 . The discrete Fréchet distance $d_F(P, Q)$ between the sequences of points P and Q is defined to be

$$d_F(P, Q) = \min\{\|L\|; L \text{ is a coupling between } P \text{ and } Q\}.$$

Intuitively, this can be defined as follows. A man is walking a dog on a leash. The man can move on the points in the sequence P , and the dog in the

sequence Q , but backtracking is not allowed. The discrete Fréchet distance $d_F(P, Q)$ is the length of the shortest leash that is sufficient for traversing both sequences. For a pair of melodies \mathbf{a}, \mathbf{b} , we define the discrete Fréchet distance $d_F(\mathbf{a}, \mathbf{b})$ to be $d_F(V(\mathbf{a}), V(\mathbf{b}))$. Furthermore taking into account the fact that any transposition of a melody does not change its essential feature, we define the transposed discrete Fréchet distance $d_F^{tr}(\mathbf{a}, \mathbf{b})$ by the following rule:

$$d_F^{tr}(\mathbf{a}, \mathbf{b}) = \min_{t \in \mathbb{Z}} d_F(\mathbf{a}, \mathbf{b} + t), \quad (3.1)$$

where $\mathbf{b} + t$ denotes the transposed melody $(b_1 + t, \dots, b_m + t)$. In an actual computation of $d_F^{tr}(\mathbf{a}, \mathbf{b})$, we can choose a bound B such that the minimum on the right hand side of (2.1) lies in $[-B, B]$.

Example 3.1. Let $\mathbf{a}_1 = (0, 2, 4, 5, 2, 2, 0)$ and $\mathbf{b}_1 = (0, 2, 5, 2, 1)$. The point sequences which correspond to these melodies are

$$V(\mathbf{a}_1) = (p_1, p_2, \dots, p_6),$$

with

$$\begin{aligned} p_1 &= (0, 2), & p_2 &= (2, 4), & p_3 &= (4, 5), & p_4 &= (5, 2), \\ p_5 &= (2, 2), & p_6 &= (2, 0), \end{aligned}$$

and

$$V(\mathbf{b}_1) = (q_1, q_2, \dots, q_4),$$

with

$$q_1 = (0, 2), \quad q_2 = (2, 5), \quad q_3 = (5, 2), \quad q_4 = (2, 1).$$

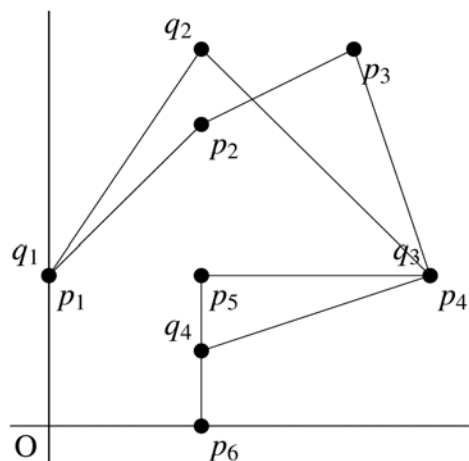


Figure 4. $M(\mathbf{a}_1)$ and $M(\mathbf{b}_1)$.

The coupling L which attains the minimum of $\|L\|$ is found to be

$$L = ((p_1, q_1), (p_2, q_2), (p_3, q_2), (p_4, q_3), (p_5, q_4), (p_6, q_4))$$

with $\|L\| = 2$. Note that when the man Paul (for P) goes to p_3 , his dog Queen (for Q) must remain at q_2 , because if Queen moves to q_3 , then $d(p_3, q_3) = \sqrt{10} > 2 = d(p_3, q_2)$. We recommend the reader to take a walk with Queen several times, then he will be convinced that the above coupling L is the best choice.

Example 3.2. Let

$$\mathbf{a}_2 = (0, 2, 4, 5, 7) = (C4, D4, E4, F4, G4),$$

$$\mathbf{b}_2 = (2, 9, 7, 6, 4) = (D4, A4, G4, F\#4, E4).$$

The discrete Fréchet distances between \mathbf{a}_2 and $\mathbf{b}_2 + t$ with $t = -5, -4, \dots, 0, 1$ are tabulated as follows:

Table 9. Discrete Fréchet distances

t	$d_F(\mathbf{a}_2, \mathbf{b}_2 + t)$
-5	8.944
-4	7.616
-3	6.325
-2	5.099
-1	6.083
0	7.280
1	8.544

It follows that $d_F^{tr}(\mathbf{a}_2, \mathbf{b}_2) = d_F(\mathbf{a}_2, \mathbf{b}_2 - 2) = 5.099$. This seems to be natural, since the melody \mathbf{a}_2 is in C major, the melody \mathbf{b}_2 in D major, and $C4 - D4 = 0 - 2 = -2$. The next example, however, shows us that the situation is not so simple.

Example 3.3. Let

$$\mathbf{a}_3 = (0, 2, 4, 5, 7) = (C4, D4, E4, F4, G4),$$

$$\mathbf{b}_3 = (0, 4, 7, 12) = (C4, E4, G4, C5).$$

The discrete Fréchet distances between \mathbf{a}_3 and $\mathbf{b}_3 + t$ with $t = -5, -4, \dots, 0, 1$ are tabulated as follows:

Table 10. Discrete Fréchet distances

t	$d_F(\mathbf{a}_3, \mathbf{b}_3 + t)$
-5	5.831
-4	4.472
-3	3.162
-2	3.000
-1	4.123
0	5.385
1	6.708

It follows that $d_F^{tr}(\mathbf{a}_3, \mathbf{b}_3) = d_F(\mathbf{a}_3, \mathbf{b}_3 - 2) = 3.000$. This time both melodies are in C major, but they require a transposition by -2 . Indeed the arithmetic mean of the entries in the melody \mathbf{a}_3 is 3.600, that for \mathbf{b}_3 is 5.750, and their difference is equal to $-2.15 \approx -2$.

In Example 3.2, the arithmetic mean of \mathbf{a}_2 is 3.600, that of \mathbf{b}_2 is 5.600, and their difference is equal to -2 . This together with Example 3.3 shows that the relevance of difference of the arithmetic means of two melodies when we compute the transposed Fréchet distance. These phenomena lead us to consider the DTFD's between a melody and its permutations. Note that in this case their arithmetic means are one and the same.

Example 3.4. Let us fix $\mathbf{a}_4 = (0, 2, 4) = (\text{C2}, \text{D4}, \text{E4})$ and let \mathbf{b}_4 run in the set of permutations of $\{0, 2, 4\}$. The following table displays the values of t for which $d_F(\mathbf{a}_4, \mathbf{b}_4 + t)$ attains the minimum:

Table 11. Transposed discrete Fréchet distances

\mathbf{a}_4	\mathbf{b}_4	t with minimum $d_F(\mathbf{a}_4, \mathbf{b}_4 + t)$	distance
(0,2,4)	(0,2,4)	0	0
(0,2,4)	(0,4,2)	0	2.828
(0,2,4)	(2,0,4)	0	2.828
(0,2,4)	(2,4,0)	1	4.243
(0,2,4)	(4,0,2)	-1	4.243
(0,2,4)	(4,2,0)	0	4.000

These examples teach us a lesson that the difference of the arithmetic means is a tentative value for us to find what value of t gives us the minimum of DFD.

3.2. Cluster analysis based on TDFD

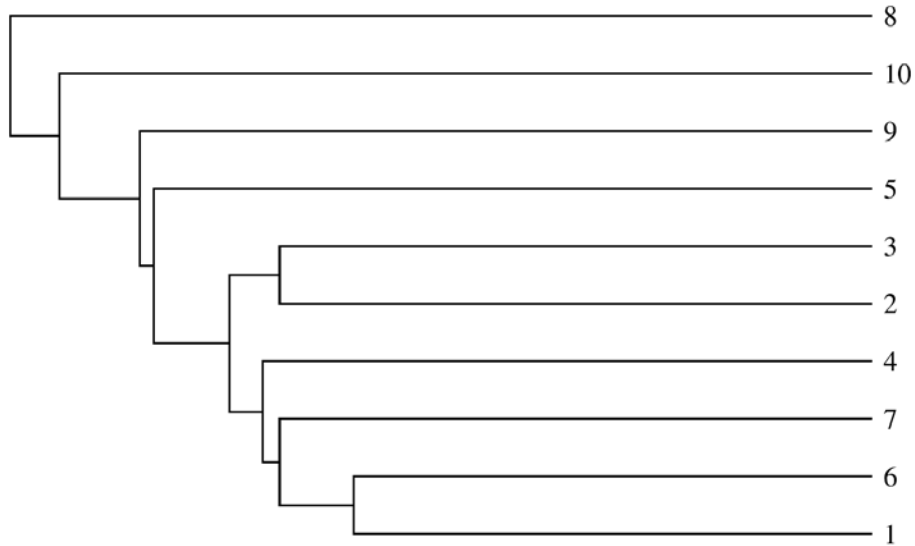
In this subsection, we analyze the cluster structure of some instances of melodies by using TDFD.

As samples we choose several national anthems. The following table shows the names of countries, their national anthems in terms of numbers, and their slopes:

Table 12. Slopes of national anthems

	name of country	national anthem	slope
1	Austria	(12,10,9,10,12,14,12,12,10,10)	0.460
2	Bulgaria	(4,9,9,11,12,11,9,4,9,9,11,12,11,9)	0.257
3	Canada	(7,10,10,3,5,7,9,10,12,5)	0.110
4	China	(7,11,14,14,16,14,11,7,14,14,14,11,7)	0.285
5	Germany	(7,9,11,9,12,11,9,6,7,16,14,12,11,9,11,7,14)	0.131
6	Hungary	(2,3,5,10,5,3,2,7,5,3,2,0,2,3)	0.359
7	Israel	(0,2,3,5,7,7,7,8,7,8,12,7,5,5,5,3,5,3,2,0,2,3,0)	0.743
8	Japan	(2,0,2,4,7,4,2,4,7,9,7,9,14,11,9,7)	0.729
9	Morocco	(10,12,10,7,8,10,3,5,7,7,8,0,7,8,5)	0.165
10	New Zealand	(7,6,7,2,11,11,9,7,4,12,2,11,9,7,6,4,2)	-0.197

For these melodies, we apply clustering by using group average method. The result is as follows:

**Figure 5.** Dendrogram for melodies 1 to 10.

One can see that the melody 1 (Austria) and the melody 6 (Hungary) is the closest pair. Furthermore the following table reveals a fascinating fact:

Table 13. DFD between “Austria” and transposed “Hungary”

t	$d_F(\text{“Austria”}, \text{“Hungary”} + t)$
4	7.211
5	5.831
6	4.472
7	3.606
8	4.123
9	5.385
10	6.708

The arithmetic mean of “Austria” is equal to 11.100 and that of “Hungary” is 3.714, and hence the difference is $7.386 \approx 7$, which coincides the value of t giving the minimum of DFD under transposition. Actually, “Austria” is in F major, and “Hungary” is in B \flat major. Hence our TDFD detects the difference of their arithmetic means as well as the difference $F4 - B\flat3 = 5 - (-2) = 7$, which is required to transpose “Hungary” to “Austria”. Moreover, though they are in different time, one is in three-four time, the other in four-four, both melodies can be played at the same time quite harmoniously.

Next, we consider how cluster structure changes if we add some other melodies to the above 10 national anthems. Let us choose “Twinkle Twinkle Little Star” as the 11th melody:

$$11 : \text{Twinkle} = (0, 0, 7, 7, 9, 9, 7, 5, 5, 4, 4, 2, 2, 0) : \text{slope} = 0.690.$$

This time the result of clustering becomes as follows:

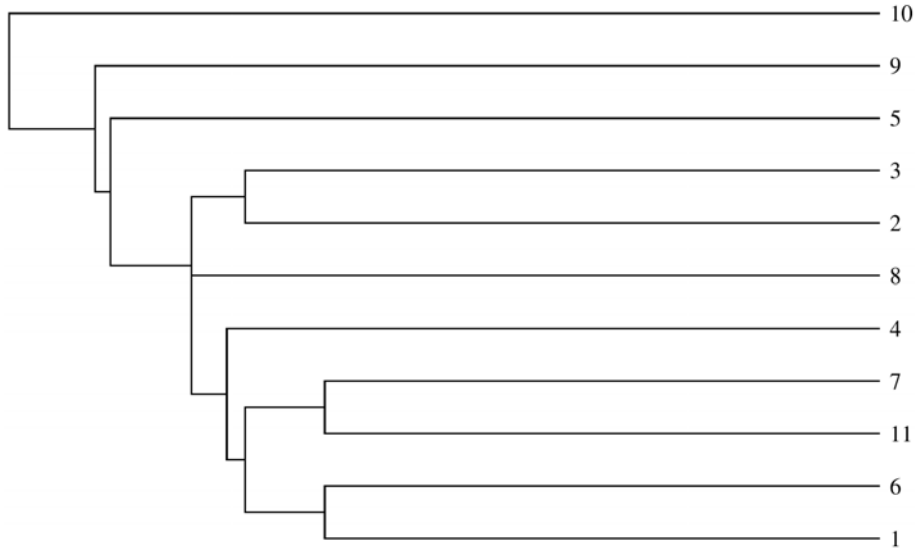


Figure 6. Dendrogram for melodies 1 to 11.

Here appears a new nearest pair (7: Israel, 11: Twinkle) with $d_F^r(7, 11) = 3.606$, which is equal to TDFD between 1 and 6. Amazingly enough, the melodies 7 and 11 can be sung harmoniously under the condition that Twinkle is transposed to C minor.

Furthermore, we add to the samples a Japanese song called “Kojo no Tsuki (Moon over the Ruined Castle)” as the 12th melody:

$$12 : \text{Kojo} = (6, 6, 11, 13, 14, 13, 11, 7, 7, 6, 5, 6) : \text{slope} = 0.762.$$

This time the result of clustering becomes as follows:

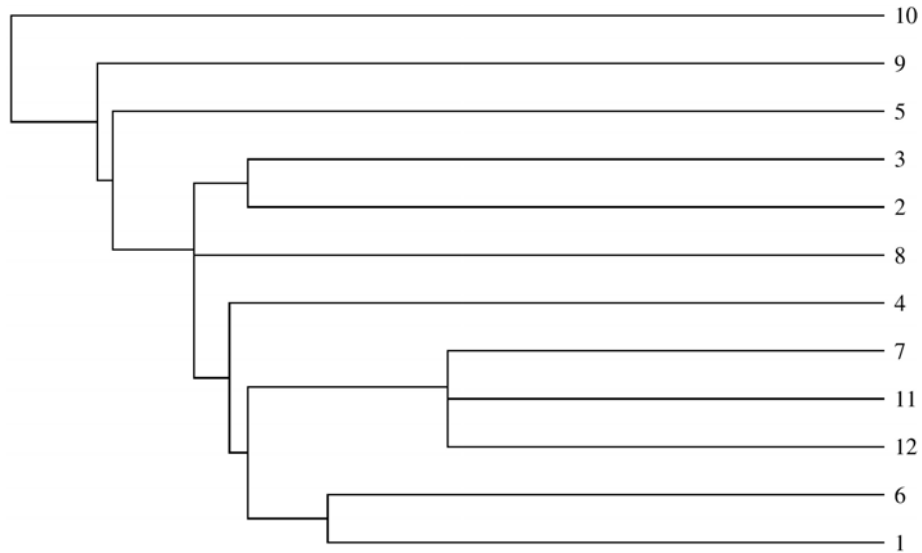


Figure 7. Dendrogram for melodies 1 to 12.

Here appears a cluster (7: Israel, 11: Twinkle, 12: Kojo) where $d_F^T(7, 12) = d_F^T(11, 12) = 2.828$. We are surprised again to find that these three melodies can be sung harmoniously if Twinkle and Kojo are transposed to C minor.

3.3. Motivating example

This subsection explains how the author came across the cluster (7: Israel, 11: Twinkle, 12: Kojo). In the fall term in 2013 he gave lessons in the discrete Fréchet distance at his university, and in a class he sent out to the students questionnaire about their most favorite musics. The following figure shows the result of clustering based on their answers:

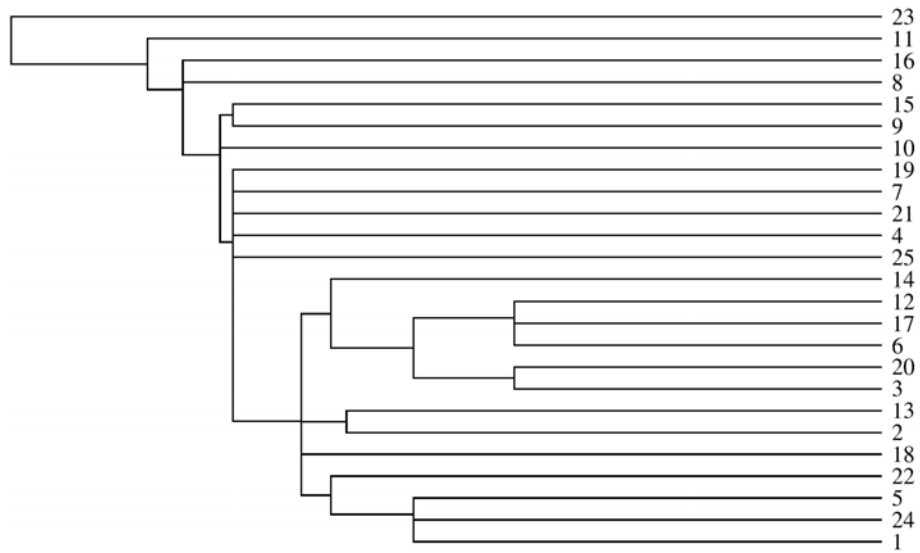


Figure 8. Dendrogram for answers to questionnaire by the author.

However their choices tended to be Japanese songs, there were some foreign ones. The following list is a part of them:

- 3: The Moldau,
- 6: Kojo,
- 8: It's a Small World,
- 10: Fantaisie-Impromptu by Chopin,
- 11: Grande valse brillante by Chopin,
- 12: National anthem of Israel,
- 13: Prelude to "Die Meistersinger" by Wagner,
- 14: Erlkönig by Shubert,
- 16: National anthem of New Zealand,
- 17: Twinkle,
- 20: Ievan Polkka,
- 25: Anchors Aweigh.

In Figure 8, one notices the cluster $\{6, 12, 17\}$, which is exactly the one investigated in the previous section. Furthermore there is another cluster $\{3, 20\}$, “The Moldau” and “Ievan Polkka”, which can be sung simultaneously too! These surprises in the result of questionnaire motivated the author to study in this article the usefulness of the transposed discrete Fréchet distance.

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