

### 解析学 III 演習 No. 1

[1] 次の2変数関数について、それぞれの極限值を求めよ。

$$(i) \lim_{(x,y) \rightarrow (1,0)} \frac{x^2 - 3xy + 2}{x^2 + y^2}.$$

$$(ii) \lim_{(x,y) \rightarrow (0,0)} \frac{5x^2y}{x^2 + y^2}.$$

$$(iii) \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}.$$

$$(iv) \lim_{(x,y) \rightarrow (0,0)} \frac{2x + y - 3}{x^2 + y^2 + 5}.$$

$$(v) \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}.$$

$$(vi) \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2y + xy^2)}{xy}.$$

[2] 次の2変数関数の極限は存在しないことを示せ。

$$(i) \lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2 + y^2}.$$

$$(ii) \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4}.$$

$$(iii) \lim_{(x,y) \rightarrow (0,0)} \frac{x + y}{\sqrt{x^2 + y^2}}.$$

[3] 次の2変数関数  $f(x, y)$  について、 $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ ,  $\lim_{y \rightarrow 0} \{ \lim_{x \rightarrow 0} f(x, y) \}$ ,

$\lim_{x \rightarrow 0} \{ \lim_{y \rightarrow 0} f(x, y) \}$  を求めよ。

$$(i) f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}.$$

$$(ii) f(x, y) = \frac{x^2 + y^2}{xy + (x - y)^2}.$$

$$(iii) f(x, y) = \frac{\sin x \sin y}{x^2 + y^2}.$$

$$(iv) f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}.$$

## 解析学 III 演習 No. 1 の解答例

[1] (i) 3.

以下必要ならば, 極座標変換  $x = r \cos \theta, y = r \sin \theta$  をする. また, 関数を  $f(x, y)$  と表す.

(ii)  $|f(x, y)| \leq |5r \sin \theta \cos^2 \theta| \leq 5r \rightarrow 0$  ( $r \rightarrow 0$ ) より, 極限值は 0.

(iii)  $|f(x, y)| = \left| \frac{r^2 \cos \theta \sin \theta}{r} \right| \leq r \rightarrow 0$  ( $r \rightarrow 0$ ).

(iv)  $-3/5$ .

(v)  $\lim_{r \rightarrow 0} \frac{\sin r^2}{r^2} = 1$ .

(vi)  $\frac{\sin xy(x+y)}{xy(x+y)}(x+y) \rightarrow 0$  ( $(x, y) \rightarrow (0, 0)$ ).

[2] (i)  $y = mx$  に沿って  $(0, 0)$  に近づくと,  $2m/(1+m^2)$ .

(ii)  $y^2 = mx$  に沿って  $(0, 0)$  に近づくと,  $m/(1+m^2)$ .

(ii)  $y = mx$  に沿って  $(0, 0)$  に近づくと,  $(1+m)x/\sqrt{1+m^2}|x|$ .

[3] (i)  $y = mx$  に沿って  $(0, 0)$  に近づくと,  $(m^2 - 1)/(m^2 + 1)$ .

$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  は存在しない.  $\lim_{y \rightarrow 0} \{ \lim_{x \rightarrow 0} f(x, y) \} = 1, \lim_{x \rightarrow 0} \{ \lim_{y \rightarrow 0} f(x, y) \} = -1$ .

(ii)  $y = mx$  に沿って  $(0, 0)$  に近づくと,  $(1+m^2)/(m+(1-m)^2)$ .

$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  は存在しない.  $\lim_{y \rightarrow 0} \{ \lim_{x \rightarrow 0} f(x, y) \} = 1, \lim_{x \rightarrow 0} \{ \lim_{y \rightarrow 0} f(x, y) \} = 1$ .

(iii)  $y = mx$  に沿って  $(0, 0)$  に近づくと,  $\frac{\sin x \sin y}{x^2 + y^2} = \left( \frac{\sin x}{x} \right) \left( \frac{\sin mx}{mx} \right) \frac{m}{1+m^2}$ .

$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  は存在しない.  $\lim_{y \rightarrow 0} \{ \lim_{x \rightarrow 0} f(x, y) \} = 0, \lim_{x \rightarrow 0} \{ \lim_{y \rightarrow 0} f(x, y) \} = 0$ .

(iv) 極座標変換  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0, \lim_{y \rightarrow 0} \{ \lim_{x \rightarrow 0} f(x, y) \} = 0, \lim_{x \rightarrow 0} \{ \lim_{y \rightarrow 0} f(x, y) \} = 0$ .

## 解析学 III 演習 No. 2

[ 1 ] 次の2変数関数  $f(x, y)$  の  $(a, b)$  における各変数に関する偏微分係数を求めよ.

(i)  $f(x, y) = \frac{1}{x^2 + y^2}, \quad (a, b) = (2, 3).$

(ii)  $f(x, y) = \log(x + y), \quad (a, b) = (2, 5).$

(iii)  $f(x, y) = e^{x^2+y^2}, \quad (a, b) = (-1, 1).$

(iv)

$$f(x, y) = \begin{cases} \frac{\sin x}{x^2+y^2} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0), \end{cases} \quad (a, b) = (0, 0)$$

(v)  $f(x, y) = \sqrt{|xy|}, \quad (a, b) = (0, 0).$

(vi)  $f(x, y) = \sqrt{x^2 + y^2}, \quad (a, b) = (0, 0).$

(vii)

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0), \end{cases} \quad (a, b) = (0, 0).$$

[ 2 ] 次の多変数関数のそれぞれの変数に関する偏導関数を求めよ.

(i)  $f(x, y) = (x^3 - 1)(y + 3).$

(ii)  $f(x, y) = \log_y x.$

(iii)  $f(x, y) = x^y.$

(iv)  $f(x, y) = e^{-y}(\cos(x + y)).$

(v)  $f(x, y) = x^y y^x.$

(vi)  $f(x, y, z) = \frac{x+y}{x-z}.$

(vii)  $f(x, y, z) = \log \sqrt{x^2 + y^2 + z^2}.$

(viii)  $f(x, y, z) = \sin^{-1}(xyz).$

解析学 III 演習 No. 2 の解答例

- [ 1 ] (i)  $f_x(2, 3) = -4/169, f_y(2, 3) = -6/169.$   
(ii)  $f_x(2, 5) = 1/7, f_y(2, 5) = 1/7.$   
(iii)  $f_x(-1, 1) = -2e^2, f_y(-1, 1) = 2e^2.$   
(iv)  $f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h}{h^3}$  は存在しない.  
 $f_y(0, 0) = 0.$   
(v)  $f_x(0, 0) = 0, f_y(0, 0) = 0.$   
(vi)  $f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h}$  は存在しない.  
 $f_y(0, 0) = \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{|k|}{k}$  は存在しない.  
(vii)  $f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0.$   
 $f_y(0, 0) = \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{0}{k} = 0.$
- [ 2 ] (i)  $f_x(x, y) = 3x^2(y + 3), f_y(x, y) = x^3 - 1.$   
(ii)  $f_x(x, y) = \frac{1}{x \log y}, f_y(x, y) = -\frac{\log x}{y(\log y)^2}.$   
(iii)  $f_x(x, y) = x^{y-1}y, f_y(x, y) = x^y \log x.$   
(iv)  $f_x(x, y) = -e^{-y} \sin(x + y), f_y(x, y) = e^{-y} \cos(x + y) - e^{-y} \sin(x + y).$   
(v)  $f_x(x, y) = yx^{y-1}y^x + x^y y^x \log y, f_y(x, y) = x^y x y^{x-1} + y^x x^y \log x.$   
(vi)  $f_x(x, y, z) = -\frac{y+z}{(x-z)^2}, f_y(x, y, z) = \frac{1}{x-z}, f_z(x, y, z) = \frac{x+y}{(x-z)^2}.$   
(vii)  $f_x(x, y, z) = \frac{x}{x^2+y^2+z^2}, f_y(x, y, z) = \frac{y}{x^2+y^2+z^2}, f_z(x, y, z) = \frac{z}{x^2+y^2+z^2}.$   
(viii)  $f_x(x, y, z) = \frac{yz}{\sqrt{1-x^2y^2z^2}}, f_y(x, y, z) = \frac{xz}{\sqrt{1-x^2y^2z^2}}, f_z(x, y, z) = \frac{xy}{\sqrt{1-x^2y^2z^2}}.$

### 解析学 III 演習 No. 3

[1] 次の2変数関数  $f(x, y)$  の点  $(a, b)$  での勾配ベクトルを求めよ.

(i)  $f(x, y) = x^3 - 4xy + 2y^5$   $(a, b) = (1, -1)$ .

(ii)  $f(x, y) = \sin(x + 2y)$   $(a, b) = (2, 3)$ .

(iii)  $f(x, y) = e^{x^2 - y^2}$   $(a, b) = (0, -1)$ .

[2] 次の3変数関数  $f(x, y, z)$  の点  $(a, b, c)$  での勾配ベクトルを求めよ.

(i)  $f(x, y, z) = 2x + 3y^2 - 4z^3 + 5xyz$   $(a, b, c) = (1, 1, 1)$ .

(ii)  $f(x, y, z) = \cos(x - 2y + z^2)$   $(a, b, c) = (\pi, \pi/2, 0)$ .

(iii)  $f(x, y, z) = \log(x^2 + y + 2z + 4)$   $(a, b, c) = (1, 2, 3)$ .

[3] 次の関数の点  $(a, b)$  または  $(a, b, c)$  におけるベクトル  $\mathbf{e}$  への方向微分係数を求めよ.

(i)  $f(x, y) = 2x + 3y - 1$ ,  $(a, b) = (1, 2)$ ,  $\mathbf{e} = (1/\sqrt{2}, 1/\sqrt{2})$ .

(ii)  $f(x, y) = x^2 + y^2 - 4$ ,  $(a, b) = (c, d)$ ,  $\mathbf{e} = (-1/\sqrt{2}, 1/\sqrt{2})$ .

(iii)  $f(x, y, z) = xy + yz + zx$ ,  $(a, b, c) = (1, 2, -1)$ ,  $\mathbf{e} = (1, 2, -3)/\sqrt{14}$ .

(iv)  $f(x, y, z) = \sin(xyz)$ ,  $(a, b, c) = (\pi, 1/2, 1/3)$ ,  $\mathbf{e} = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ .

(v)  $f(x, y, z) = x - \sqrt{y^2 + z^2}$ ,  $(a, b, c) = (-2, 1, -3)$ ,  $\mathbf{e} = (2, -1, 4)/\sqrt{21}$ .

### 解析学 III 演習 No. 3 の解答例

[ 1 ] (i)  $(7, 6)$ .

(ii)  $(\cos 8, 2 \cos 8)$ .

(iii)  $(0, 2e^{-1})$ .

[ 2 ] (i)  $(7, 11, -7)$ .

(ii)  $(0, 0, 0)$ .

(iii)  $(2/13, 1/13, 2/13)$ .

[ 3 ] (i)  $5/\sqrt{2}$ .

(ii)  $(-2c + 2d)/\sqrt{2}$ .

(iii)  $-8/\sqrt{14}$ .

(iv)  $(1 + \pi)/12$ .

(v)  $2/\sqrt{21} + 13/\sqrt{210}$ .

### 解析学 III 演習 No. 4

[1]  $z = f(x, y)$  が 2 階連続微分可能で,  $x = a + ut, y = b + vt$  であるとき,

$$\frac{dz}{dt} = uf_x + vf_y, \quad \frac{d^2z}{dt^2} = u^2 f_{xx} + 2uv f_{xy} + v^2 f_{yy}$$

が成り立つことを証明せよ. ただし,  $a, b, u, v \in \mathbb{R}$  は定数.

[2]  $z = f(x, y)$  が 2 階連続微分可能で,  $x = r \cos \theta, y = r \sin \theta$  であるとき, 次を示せ.

(i)  $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = r \frac{\partial z}{\partial r}.$

(ii)  $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2.$

(iii)  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \left(\frac{\partial z}{\partial r} + \frac{1}{r} \frac{\partial^2 z}{\partial \theta^2}\right).$

解析学 III 演習 No. 4 の解答例

[ 1 ] (i)

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = u f_x + v f_y.$$

(ii)

$$\begin{aligned} \frac{d^2 z}{dt^2} &= \frac{\partial(u f_x + v f_y)}{\partial x} \frac{dx}{dt} + \frac{\partial(u f_x + v f_y)}{\partial y} \frac{dy}{dt} \\ &= (u f_{xx} + v f_{yx})u + (u f_{xy} + v f_{yy})v \\ &= u^2 f_{xx} + 2uv f_{xy} + v^2 f_{yy}. \end{aligned}$$

[ 2 ] (i)

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y}.$$

故に,

$$r \frac{\partial z}{\partial r} = r \cos \theta \frac{\partial z}{\partial x} + r \sin \theta \frac{\partial z}{\partial y} = x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}.$$

(ii)

$$\left( \frac{\partial z}{\partial r} \right)^2 = \cos^2 \theta \left( \frac{\partial z}{\partial x} \right)^2 + \sin^2 \theta \left( \frac{\partial z}{\partial y} \right)^2 + 2 \cos \theta \sin \theta \frac{\partial z}{\partial x} \frac{\partial z}{\partial y},$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = (-r \sin \theta) \frac{\partial z}{\partial x} + (r \cos \theta) \frac{\partial z}{\partial y}.$$

故に,

$$\frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2 = \sin^2 \theta \left( \frac{\partial z}{\partial x} \right)^2 + \cos^2 \theta \left( \frac{\partial z}{\partial y} \right)^2 - 2 \cos \theta \sin \theta \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}.$$

よって,

$$\left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2 = \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2.$$

(iii)

$$\begin{aligned} \frac{\partial^2 z}{\partial r^2} &= \cos \theta \frac{\partial}{\partial x} \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \sin \theta \frac{\partial}{\partial x} \frac{\partial z}{\partial y} \frac{\partial x}{\partial r} \\ &\quad + \cos \theta \frac{\partial}{\partial y} \frac{\partial z}{\partial x} \frac{\partial y}{\partial r} + \sin \theta \frac{\partial}{\partial y} \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} \\ &= \cos^2 \theta \frac{\partial^2 z}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 z}{\partial y^2}. \end{aligned}$$



$$\begin{aligned}
\frac{\partial^2 z}{\partial \theta^2} &= (-r \cos \theta) \frac{\partial z}{\partial x} + (-r \sin \theta) \frac{\partial}{\partial \theta} \frac{\partial z}{\partial x} + (-r \sin \theta) \frac{\partial z}{\partial y} + (r \cos \theta) \frac{\partial}{\partial \theta} \frac{\partial z}{\partial y} \\
&= (-r \cos \theta) \frac{\partial z}{\partial x} + (-r \sin \theta) \left( \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial \theta} \right) \\
&\quad + (-r \sin \theta) \frac{\partial z}{\partial y} + (r \cos \theta) \left( \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial \theta} \right) \\
&= (r^2 \sin^2 \theta) \frac{\partial^2 z}{\partial x^2} + (-2r^2 \sin \theta \cos \theta) \frac{\partial^2 z}{\partial x \partial y} \\
&\quad + (r^2 \cos^2 \theta) \frac{\partial^2 z}{\partial y^2} - r \left( \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \right).
\end{aligned}$$

故に,

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \left( \frac{\partial z}{\partial r} + \frac{1}{r} \frac{\partial^2 z}{\partial \theta^2} \right) = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.$$

### 解析学 III 演習 No. 5

[1] 次の2変数関数について,

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y)$$

を求めよ.

(i)  $f(x, y) = \sin x \sin y$ .

(ii)  $f(x, y) = (x - y)/(x + y)$ .

[2] 次の3変数関数について,

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^2 f(x, y, z)$$

を求めよ.

(i)  $f(x, y, z) = x^3 + y^3 + z^3$ .

(ii)  $f(x, y, z) = xze^{x^2-y^2}$ .

(iii)  $f(x, y, z) = \cos(x - yz)$ .

[3] 次の2変数関数を点  $(0, 0)$  について, 2次のテイラー多項式  $P_2$  とその剰余項  $R$  を求めよ.

(i)  $f(x, y) = \cos(x + y)$ .

(ii)  $f(x, y) = \cos x \sin y$ .

(iii)  $f(x, y) = ye^{x-y}$ .

(iv)  $f(x, y) = \sin(x + y)$ .

(v)  $f(x, y) = \sin x + \cos y$ .

### 解析学 III 演習 No. 5 の解答例

[ 1 ] (i)  $h^2(-\sin x \sin y) + 2hk \cos x \cos y + k^2(-\sin x \sin y)$ .  
(ii)  $h^2 \frac{(-4y)}{(x+y)^3} + 2hk \frac{2(x-y)}{(x+y)^3} + k^2 \frac{4x}{(x+y)^3}$ .

[ 2 ] (i)  $6(h^2x + k^2y + l^2z)$ .  
(ii)

$$2xz(3 + 2x^2)e^{x^2-y^2}h^2 - 4yz(1 + 2x^2)e^{x^2-y^2}hk + 2xz(2y^2 - 1)e^{x^2-y^2}k^2 \\ + 2(1 + 2x^2)e^{x^2-y^2}hl - 4xye^{x^2-y^2}kl.$$

(iii)

$$-\cos(x - yz)h^2 + 2z \cos(x - yz)hk - z^2 \cos(x - yz)k^2 \\ + 2y \cos(x - yz)hl - y^2 \cos(x - yz)l^2 + 2(\sin(x - yz) - yz \cos(x - yz))kl.$$

[ 3 ] (i)  $P_2 = 1 - \frac{1}{2}(x^2 + 2xy + y^2)$ .  $R = \frac{1}{6}(x^2 + 3x^2y + 3xy^2 + y^3) \sin(\theta x + \theta y)$ .

(ii)  $P_2 = y$ .  $R = \frac{1}{6}(x^3 \sin \theta x \sin \theta y - 3x^2y \cos \theta x \cos \theta y + 3xy^2 \sin \theta x \sin \theta y - y^3 \cos \theta x \cos \theta y)$ .

(iii)  $P_2 = y + xy - y^2$ .  $R = \frac{1}{6}\{x^3\theta y + 3x^2y(1 - \theta y) + 3xy^2(-2 + \theta y) + y^3(3 - \theta y)\}e^{\theta x - \theta y}$ .

(iv)  $P_2 = x + y$ .  $R = -\frac{1}{6}(x^3 + 3x^2y + 3xy^2 + y^3) \cos(\theta x + \theta y)$ .

(v)  $P_2 = 1 + x - \frac{y^2}{2}$ .  $R = \frac{1}{6}(-x^3 \cos \theta x + y^3 \sin \theta y)$

### 解析学 III 演習 No. 6

[1] 次の  $\mathbb{R}^2$  で定義された 2 変数関数について, 極大点, 極小点, 鞍点を求めよ. また, 最大値, 最小値が存在する場合はそれらを求めよ.

(i)  $f(x, y) = 5 + x + y - x^2 - y^2$ .

(ii)  $f(x, y) = x^2 + xy$ .

(iii)  $f(x, y) = x^2 - xy + y^2 + 3x - y + 4$ .

(iv)  $f(x, y) = 3xy - x^4 - y^4 + 2$ .

(v)  $f(x, y) = \frac{1}{x^2 + y^2 + 1}$ .

(vi)  $f(x, y) = \frac{1}{x} + 2xy + \frac{1}{y}$ .

(vii)  $f(x, y) = e^{-x^2+y^2-2y}$ .

(viii)  $f(x, y) = xy \log(x^2 + y^2)$ .

## 解析学 III 演習 No. 6 の解答例

[1] (i)  $f_x = 1 - 2x = 0, f_y = 1 - 2y = 0$  を解いて,  $x = y = 1/2$ .  
 $f_{xx} = -2, f_{xy} = 0, f_{yy} = -2$ .

$$H_f(1/2, 1/2) = f_{xx}(1/2, 1/2)f_{yy}(1/2, 1/2) - f_{xy}(1/2, 1/2)^2 = 4 > 0.$$

$f_{xx}(1/2, 1/2) = -2 < 0$  だから,  $(x, y) = (1/2, 1/2)$  で極大. 唯一の極大点であるから, そこで最大となり, 最大値は  $11/2$ .

(ii)  $f_x = 2x + y = 0, f_y = x = 0$  を解いて,  $x = y = 0$ .  $f_{xx} = 2, f_{xy} = 1, f_{yy} = 0$ .

$$H_f(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = -1 < 0$$

だから,  $(x, y) = (0, 0)$  は鞍点.

(iii)  $f_x = 2x - y + 3 = 0, f_y = -x + 2y - 1 = 0$  より,  $x = -5/3, y = -1/3$ .  
 $f_{xx} = 2, f_{xy} = -1, f_{yy} = 2$ .  $H_f(-5/3, -1/3) = 3 > 0$ .  $f_{xx}(-5/3, -1/3) = 2 > 0$  だから,  $(x, y) = (-5/3, -1/3)$  で極小. 唯一の極小だから, 最小でもあり, 最小値は  $f(-5/3, -1/3) = 5/3$ .

(iv)  $f_x = 3y - 4x^3 = 0, f_y = 3x - 4y^3 = 0$  より,  $(x, y) = (0, 0), (\sqrt{3}/2, \sqrt{3}/3), (-\sqrt{3}/2, -\sqrt{3}/2)$ .  $f_{xx} = -12x^2, f_{xy} = 3, f_{yy} = -12y^2$ .

$H_f(0, 0) = -9 < 0$  だから,  $(x, y) = (0, 0)$  は鞍点.

$H_f(\sqrt{3}/2, \sqrt{3}/2) = 72 > 0$ .  $f_{xx}(\sqrt{3}/2, \sqrt{3}/2) = -9 < 0$  だから,  $(\sqrt{3}/2, \sqrt{3}/2)$  で極大.

$H_f(-\sqrt{3}/2, -\sqrt{3}/2) = 72 > 0$ .  $f_{xx}(-\sqrt{3}/2, -\sqrt{3}/2) = -9 < 0$  だから,  $(-\sqrt{3}/2, -\sqrt{3}/2)$  で極大.

(v)  $f_x = (-1)(x^2 + y^2 + 1)^{-2}2x = 0, f_y = (-1)(x^2 + y^2 + 1)^{-2}2y = 0$  より,  $(x, y) = (0, 0)$ .  $f_{xx} = 2(x^2 + y^2 + 1)^{-3}4x^2 - 2(x^2 + y^2 + 1)^{-2}, f_{yy} = 2(x^2 + y^2 + 1)^{-3}4y^2 - 2(x^2 + y^2 + 1)^{-2}, f_{xy} = 2(x^2 + y^2 + 1)^{-3}4xy$ .

$H_f(0, 0) = 4 > 0, f_{xx}(0, 0) = -2 < 0$  だから,  $(x, y) = (0, 0)$  で極大. 唯一の極大点であるから最大で, 最大値は  $f(0, 0) = 1$ .

(vi)  $f_x = \frac{-1}{x^2} + 2y = 0, f_y = 2x + \frac{-1}{y^2} = 0$  より,  $2x^2y = 1, 2xy^2 = 1$ .  $x, y > 0$  より  $x = y = 1/\sqrt[3]{2}$ .  $f_{xx} = 2/x^3, f_{yy} = 2/y^3, f_{xy} = 2$ .  $H_f(1/\sqrt[3]{2}, 1/\sqrt[3]{2}) = 12 > 0$ .  $f_{xx}(1/\sqrt[3]{2}, 1/\sqrt[3]{2}) = 4 > 0$  だから,  $(x, y) = (1/\sqrt[3]{2}, 1/\sqrt[3]{2})$  で極小. 唯一の極小点であるから, 最小. 最小値は,  $3\sqrt[3]{2}$ .

(vii)  $f_x = -2xe^{-x^2+y^2-2y} = 0, f_y = (2y-2)e^{-x^2+y^2-2y} = 0$  より,  $x = 0, y = 1$ .  $f_{xx} = -2e^{-x^2+y^2-2y} + 4x^2e^{-x^2+y^2-2y}, f_{yy} = 2e^{-x^2+y^2-2y} + (2y-2)^2e^{-x^2+y^2-2y}, f_{xy} = -2x(2y-2)e^{-x^2+y^2-2y}$ .

$H_f(0, 1) = -4e^{-2} < 0$ . よって,  $(0, 1)$  は鞍点.

(viii)  $f_x = y \log(x^2 + y^2) + \frac{2x^2y}{x^2+y^2} = 0, f_y = x \log(x^2 + y^2) + \frac{2xy^2}{x^2+y^2} = 0$  より,

$$(x, y) = (\pm 1, 0), (0, \pm 1), (1/\sqrt{2e}, 1/\sqrt{2e}), (1/\sqrt{2e}, -1/\sqrt{2e}), \\ (-1/\sqrt{2e}, 1/\sqrt{2e}), (-1/\sqrt{2e}, -1/\sqrt{2e}).$$

$$f_{xx} = \frac{2x^3y+6xy^3}{(x^2+y^2)^2}, f_{yy} = \frac{6x^3y+2xy^3}{(x^2+y^2)^2}, f_{xy} = \log(x^2 + y^2) + \frac{2(x^4+y^4)}{(x^2+y^2)^2}.$$

$H_f(\pm 1, 0) = -4 < 0, H_f(0, \pm 1) = -4 < 0$ .  $(\pm 1, 0), (0, \pm 1)$  では鞍点.

$H_f(\pm 1/\sqrt{2e}, \pm 1/\sqrt{2e}) = 4 > 0$ .  $f_{xx}(\pm 1/\sqrt{2e}, \pm 1/\sqrt{2e}) = 2 > 0$  (復号同順). よって,  $(\pm 1/\sqrt{2e}, \pm 1/\sqrt{2e})$  で極小.

$H_f(\pm 1/\sqrt{2e}, \mp 1/\sqrt{2e}) = 4 > 0$ .  $f_{xx}(\pm 1/\sqrt{2e}, \mp 1/\sqrt{2e}) = -2 < 0$  (復号同順). よって,  $(\pm 1/\sqrt{2e}, \mp 1/\sqrt{2e})$  で極大.

### 解析学 III 演習 No. 7

[1]  $y$  が  $x$  の関数であり, 次が成り立つとき,  $y$  の極値を求めよ.

(i)  $xy^2 - x^2y = 16$ .

(ii)  $x^3 + y^3 - x - y = 0$ .

(iii)  $x^3 - 3axy + y^3 = 0$  ( $a > 0$ ).

### 解析学 III 演習 No. 7 解答例

[1] (i)  $F = xy^2 - x^2y - 16 = 0, F_x = y^2 - 2xy = 0$  をみたす点は  $(x, y) = (2, 4)$  である. このとき, 陰関数を  $y = f(x)$  とし,

$$f''(2) = -\frac{F_{xx}(2, 4)}{F_y(2, 4)} = \frac{2}{3} > 0$$

よって  $y = f(x)$  は  $x = 2$  で極小値 4 をとる.

(ii)  $F = x^3 + y^3 - x - y = (x+y)(x^2 - xy + y^2 - 1) = 0, F_x = 3x^2 - 1 = 0$  をみたす点は  $(x, y) = (1/\sqrt{3}, -1/\sqrt{3}), (1/\sqrt{3}, 2/\sqrt{3}), (-1/\sqrt{3}, 1/\sqrt{3}), (-1/\sqrt{3}, -2/\sqrt{3})$  である. このとき,  $(1/\sqrt{3}, -\sqrt{3}), (-1/\sqrt{3}, 1/\sqrt{3})$  では,  $F_y = 0$  となるので除外する.

$$f''(1/\sqrt{3}) = -\frac{F_{xx}(1/\sqrt{3}, 2/\sqrt{3})}{F_y(1/\sqrt{3}, 2/\sqrt{3})} = -2/\sqrt{3} < 0$$

よって  $y$  は  $x = 1/\sqrt{3}$  で極大値  $2/\sqrt{3}$  をとる.

$$f''(-\sqrt{3}) = -\frac{F_{xx}(-1/\sqrt{3}, -2/\sqrt{3})}{F_y(-1/\sqrt{3}, -2/\sqrt{3})} = 2/\sqrt{3} > 0$$

よって  $y$  は  $x = -1/\sqrt{3}$  で極小値  $-2/\sqrt{3}$  をとる.

(iii)  $F = x^3 - 3axy + y^3 = 0, F_x = 3x^2 - 3ay = 0$  より,  $(x, y) = (0, 0), (2^{1/3}a, 2^{2/3}a)$ . 原点では,  $F_y = 0$  となる.

$$-\frac{F_{xx}(2^{1/3}, 2^{2/3})}{F_y(2^{1/3}, 2^{2/3})} = \frac{-2}{a} < 0.$$

よって,  $x = 2^{1/3}$  で極大値  $2^{2/3}$  をとる.



### 解析学 III 演習 No. 8

[1] 次の重積分を累次積分により求めよ. ただし,  $\iint_{[a,b] \times [c,d]} f(x,y) dx dy$

を  $\int_c^d \int_a^b f(x,y) dx dy$  と書く.

(i)  $\int_1^2 \int_0^1 (1+xy) dx dy$

(ii)  $\int_0^1 \int_1^2 (1+xy) dy dx$

(iii)  $\int_1^2 \int_0^1 dx dy$

(iv)  $\int_4^7 \int_{-3}^1 dy dx$

(v)  $\int_1^2 \int_{-1}^1 e^{u+v} du dv$

(vi)  $\int_{-1}^1 \int_1^2 e^{u+v} du dv$

(vii)  $\int_1^3 \int_1^2 2y \log x dy dx$

(viii)  $\int_0^{\pi/2} \int_0^{\pi/2} (\cos(x+t)) dx dt$

(ix)  $\int_{\pi}^{2\pi} \int_0^{\pi} (\sin x + \cos y) dx dy$

(x)  $\int_0^2 \int_0^{\pi} \cos x dy dx$

### 解析学 III 演習 No. 8 解答例

[ 1 ] (i)  $7/4$ .

(ii)  $7/4$ .

(iii)  $1$ .

(iv)  $12$ .

(v)  $e^3 - e^2 - e + 1$ .

(vi)  $e^3 - e^2 - e + 1$ .

(vii)  $9 \log 3 - 6$ .

(viii)  $0$ .

(ix)  $2\pi$ .

(x)  $\pi \sin 2$ .

### 解析学 III 演習 No. 9

[1] 次の重積分を求めよ.

$$(i) \int_0^2 \int_x^{x^2} dy dx.$$

$$(ii) \int_0^2 \int_0^x xy dy dx.$$

$$(iii) \int_0^\pi \int_0^x x \sin y dy dx.$$

$$(iv) \int_0^1 \int_x^{x^2} (2x - y) dy dx.$$

$$(v) \int_1^2 \int_0^{1/y} xe^{xy} dx dy.$$

$$(vi) \int_0^1 \int_{y^2}^y \sqrt{xy} dx dy.$$

$$(vii) \int_0^1 \int_0^y x \sqrt{y^2 - x^2} dx dy.$$

$$(viii) \int_1^e \int_1^{\log y} e^x dx dy.$$

$$(ix) \int_0^\pi \int_0^{\cos y} x \sin y dx dy.$$

### 解析学 III 演習 No. 9 解答例

- [ 1 ] (i)  $2/3$ .  
(ii)  $2$ .  
(iii)  $\pi^2/2 + 2$ .  
(iv)  $-1/10$ .  
(v)  $1/2$ .  
(vi)  $2/27$ .  
(vii)  $1/12$ .  
(viii)  $-e^2/2 + e - 1/2$ .  
(ix)  $1/3$ .

### 解析学 III 演習 No. 10

[1]  $f(x, y) = (x + y)^p$  の正方形  $G = \{(x, y); a \leq x \leq b, a \leq y \leq b\}$  での重積分を求めよ。ただし,  $a > 0, p \neq -2, p \neq -1$  とする。

[2]  $G$  を直線  $x + y = 2$  と放物線  $y = x^2$  とで囲まれる領域とする。このとき,

$$I = \iint_G \sqrt{y - x^2} dx dy$$

を求めよ。ただし,  $\int_0^{\pi/2} \cos^4 \theta d\theta = \frac{3}{16}\pi$  は使ってよい。

[3]  $p > -2, p \neq -1, G = \{(x, y); 0 < x \leq 1, 0 < y \leq 1\}$  のとき, 広義積分

$$\iint_G (x + y)^p dx dy$$

を求めよ。

解析学 III 演習 No. 10 解答例

[1]

$$\begin{aligned}
 \iint_G (x+y)^p dx dy &= \int_a^b dx \int_a^b (x+y)^p dy \\
 &= \int_a^b \left[ \frac{(x+y)^{p+1}}{p+1} \right]_{y=a}^{y=b} dx \\
 &= \frac{1}{p+1} \int_a^b \{(x+b)^{p+1} - (x+a)^{p+1}\} dx \\
 &= \frac{1}{(p+1)(p+2)} \{(2b)^{p+2} + (2a)^{p+2} - 2(a+b)^{p+2}\}.
 \end{aligned}$$

[2]

$$\begin{aligned}
 I &= \int_{-2}^1 dx \int_{x^2}^{2-x} \sqrt{y-x^2} dy \\
 &= \frac{2}{3} \int_{-2}^1 (2-x-x^2)^{3/2} dx \\
 &= \frac{2}{3} \int_{-2}^1 \left( \left(\frac{3}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2 \right)^{3/2} dx \\
 &= \frac{2}{3} \int_{-3/2}^{3/2} \left( \left(\frac{3}{2}\right)^2 - t^2 \right)^{3/2} dt \quad (t = x + \frac{1}{2}) \\
 &= \frac{2}{3} \left(\frac{3}{2}\right)^4 \int_{-1}^1 (1-s^2)^{3/2} ds \\
 &= \frac{2}{3} \left(\frac{3}{2}\right)^4 2 \int_0^1 (1-s^2)^{3/2} ds \quad (\text{被積分関数は偶関数}) \\
 &= \frac{4}{3} \left(\frac{3}{2}\right)^4 \int_0^{\pi/2} \cos^4 \theta d\theta \quad (s = \sin \theta) \\
 &= \frac{81}{64} \pi.
 \end{aligned}$$

[3]  $G_n = \{(x, y); 1/n \leq x \leq 1, 1/n \leq y \leq 1\}$  は  $G$  に対する一つの近似増加列である. [1] より

$$\iint_{G_n} (x+y)^p dx dy = \frac{1}{(p+1)(p+2)} \left\{ 2^{p+2} + \left(\frac{2}{n}\right)^{p+2} - 2\left(\frac{n+1}{n}\right)^{p+2} \right\}.$$

$$n \rightarrow \infty \text{ として, } \iint_G (x+y)^p dx dy = \frac{2^{p+2}-2}{(p+1)(p+2)}.$$