解析学 III 演習 No.　1

［１］ 次の２変数関数について，それぞれの極限値を求めよ．

(i) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x^2-3xy+2}{x^2+y^2}$.

(ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$.

(iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

(iv) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x+y-3}{x^2+y^2+5}$.

(v) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$.

(vi) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2y+xy^2)}{xy}$.

［２］ 次の２変数関数の極限は存在しないことを示せ．

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$.

(ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$.

(iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{\sqrt{x^2+y^2}}$.

［３］次の２変数関数 $f(x,y)$ について， $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$, $\displaystyle\lim_{y\to0}\{\lim_{x\to0} f(x,y)\}$, $\displaystyle\lim_{x\to0}\{\lim_{y\to0} f(x,y)\}$ を求めよ．

(i) $f(x,y) = \dfrac{y^2-x^2}{y^2+x^2}$.

(ii) $f(x,y) = \dfrac{x^2+y^2}{xy+(x-y)^2}$.

(iii) $f(x,y) = \dfrac{\sin x \sin y}{x^2+y^2}$.

(iv) $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$.

解析学 III 演習 No. １ の解答例

［１］ (i) 3.
以下必要ならば，極座標変換 $x = r\cos\theta, y = r\sin\theta$ をする．また，関数を $f(x,y)$ と表す．
(ii) $|f(x,y)| \le |5r\sin\theta\cos^2\theta| \le 5r \to 0$ $(r \to 0)$ より，極限値は 0.
(iii) $|f(x,y)| = |\frac{r^2\cos\theta\sin\theta}{r}| \le r \to 0$ $(r \to 0)$.
(iv) $-3/5$.
(v) $\lim_{r\to 0}\frac{\sin r^2}{r^2} = 1$.
(vi) $\frac{\sin xy(x+y)}{xy(x+y)}(x+y) \to 0$ $(x,y) \to (0,0)$.

［２］ (i) $y = mx$ に沿って $(0,0)$ に近づくと，$2m/(1+m^2)$.
(ii) $y^2 = mx$ に沿って $(0,0)$ に近づくと，$m/(1+m^2)$.
(ii) $y = mx$ に沿って $(0,0)$ に近づくと，$(1+m)x/\sqrt{1+m^2}|x|$.

［３］ (i) $y = mx$ に沿って $(0,0)$ に近づくと，$(m^2-1)/(m^2+1)$. $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ は存在しない． $\displaystyle\lim_{y\to 0}\{\lim_{x\to 0} f(x,y)\} = 1, \lim_{x\to 0}\{\lim_{y\to 0} f(x,y)\} = -1$.
(ii) $y = mx$ に沿って $(0,0)$ に近づくと，$(1+m^2)/(m+(1-m)^2)$. $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ は存在しない． $\displaystyle\lim_{y\to 0}\{\lim_{x\to 0} f(x,y)\} = 1, \lim_{x\to 0}\{\lim_{y\to 0} f(x,y)\} = 1$.
(iii) $y = mx$ に沿って $(0,0)$ に近づくと，$\frac{\sin x\sin y}{x^2+y^2} = (\frac{\sin x}{x})(\frac{\sin mx}{mx})\frac{m}{1+m^2}$. $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ は存在しない． $\displaystyle\lim_{y\to 0}\{\lim_{x\to 0} f(x,y)\} = 0, \lim_{x\to 0}\{\lim_{y\to 0} f(x,y)\} = 0$.
(iv) 極座標変換 $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$. $\displaystyle\lim_{y\to 0}\{\lim_{x\to 0} f(x,y)\} = 0, \lim_{x\to 0}\{\lim_{y\to 0} f(x,y)\} = 0$.

解析学 III 演習 No.　2

［１］ 次の２変数関数 $f(x,y)$ の (a,b) における各変数に関する偏微分係数を求めよ．

(i) $f(x,y)=\dfrac{1}{x^2+y^2}, \quad (a,b)=(2,3).$

(ii) $f(x,y)=\log(x+y), \quad (a,b)=(2,5).$

(iii) $f(x,y)=e^{x^2+y^2}, \quad (a,b)=(-1,1).$

(iv)

$$f(x,y)=\begin{cases} \frac{\sin x}{x^2+y^2} & (x,y)\neq(0,0), \\ 0 & (x,y)=(0,0), \end{cases} \qquad (a,b)=(0,0)$$

(v) $f(x,y)=\sqrt{|xy|}, \quad (a,b)=(0,0).$

(vi) $f(x,y)=\sqrt{x^2+y^2}, \quad (a,b)=(0,0).$

(vii)

$$f(x,y)=\begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0), \\ 0 & (x,y)=(0,0), \end{cases} \qquad (a,b)=(0,0).$$

［２］ 次の多変数関数のそれぞれの変数に関する偏導関数を求めよ．

(i) $f(x,y)=(x^3-1)(y+3).$
(ii) $f(x,y)=\log_y x.$
(iii) $f(x,y)=x^y.$
(iv) $f(x,y)=e^{-y}(\cos(x+y)).$
(v) $f(x,y)=x^y y^x.$
(vi) $f(x,y,z)=\frac{x+y}{x-z}.$
(vii) $f(x,y,z)=\log\sqrt{x^2+y^2+z^2}.$
(viii) $f(x,y,z)=\sin^{-1}(xyz).$

解析学 III 演習 No. 2 の解答例

［1］ (i) $f_x(2,3)=-4/169, f_y(2,3)=-6/169.$

(ii) $f_x(2,5)=1/7, f_y(2,5)=1/7.$

(iii) $f_x(-1,1)=-2e^2, f_y(-1,1)=2e^2.$

(iv) $f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{h}{h^3}$ は存在しない.

$f_y(0,0)=0.$

(v) $f_x(0,0)=0, f_y(0,0)=0.$

(vi) $f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{|h|}{h}$ は存在しない.

$f_y(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{|k|}{k}$ は存在しない.

(vii) $f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0}{h}=0.$

$f_y(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0}{k}=0.$

［2］ (i) $f_x(x,y)=3x^2(y+3), f_y(x,y)=x^3-1.$

(ii) $f_x(x,y)=\frac{1}{x\log y}, f_y(x,y)=-\frac{\log x}{y(\log y)^2}.$

(iii) $f_x(x,y)=x^{y-1}y, f_y(x,y)=x^y\log x.$

(iv) $f_x(x,y)=-e^{-y}\sin(x+y), f_y(x,y)=e^{-y}\cos(x+y)-e^{-y}\sin(x+y).$

(v) $f_x(x,y)=yx^{y-1}y^x+x^yy^x\log y, f_y(x,y)=x^yxy^{x-1}+y^xx^y\log x.$

(vi) $f_x(x,y,z)=-\frac{y+z}{(x-z)^2}, f_y(x,y,z)=\frac{1}{x-z}, f_z(x,y,z)=\frac{x+y}{(x-z)^2}.$

(vii) $f_x(x,y,z)=\frac{x}{x^2+y^2+z^2}, f_y(x,y,z)=\frac{y}{x^2+y^2+z^2}, f_z(x,y,z)=\frac{z}{x^2+y^2+z^2}.$

(viii) $f_x(x,y,z)=\frac{yz}{\sqrt{1-x^2y^2z^2}}, f_y(x,y,z)=\frac{xz}{\sqrt{1-x^2y^2z^2}} f_z(x,y,z)=\frac{xy}{\sqrt{1-x^2y^2z^2}}.$

解析学 III 演習 No. ３

［１］ 次の２変数関数 $f(x,y)$ の点 (a,b) での勾配ベクトルを求めよ．

(i) $f(x,y) = x^3 - 4xy + 2y^5 \quad (a,b) = (1,-1)$.

(ii) $f(x,y) = \sin(x+2y) \quad (a,b) = (2,3)$.

(iii) $f(x,y) = e^{x^2-y^2} \quad (a,b) = (0,-1)$.

［２］ 次の３変数関数 $f(x,y.z)$ の点 (a,b,c) での勾配ベクトルを求めよ．

(i) $f(x,y,z) = 2x + 3y^2 - 4z^3 + 5xyz \quad (a,b,c) = (1,1,1)$.

(ii) $f(x,y,z) = \cos(x - 2y + z^2) \quad (a,b,c) = (\pi, \pi/2, 0)$.

(iii) $f(x,y,z) = \log(x^2 + y + 2z + 4) \quad (a,b,c) = (1,2,3)$.

［３］ 次の関数の点 (a,b) または (a,b,c) におけるベクトル $\boldsymbol{e}$ への方向微分係数を求めよ．

(i) $f(x,y) = 2x + 3y - 1, \quad (a,b) = (1,2), \boldsymbol{e} = (1/\sqrt{2}, 1/\sqrt{2})$.

(ii) $f(x,y) = x^2 + y^2 - 4, \quad (a,b) = (c,d), \boldsymbol{e} = (-1/\sqrt{2}, 1/\sqrt{2})$.

(iii) $f(x,y,z) = xy + yz + zx, \quad (a,b,c) = (1,2,-1), \boldsymbol{e} = (1,2,-3)/\sqrt{14}$.

(iv) $f(x,y,z) = \sin(xyz), \quad (a,b,c) = (\pi, 1/2, 1/3), \boldsymbol{e} = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$.

(v) $f(x,y,z) = x - \sqrt{y^2+z^2}, \quad (a,b,c) = (-2,1,-3), \boldsymbol{e} = (2,-1,4)/\sqrt{21}$.

解析学 III 演習 No.　３ の解答例

［１］ (i) $(7, 6)$.

(ii) $(\cos 8, 2\cos 8)$.

(iii) $(0, 2e^{-1})$.

［２］ (i) $(7, 11, -7)$.

(ii) $(0, 0, 0)$.

(iii) $(2/13, 1/13, 2/13)$.

［３］ (i) $5/\sqrt{2}$.

(ii) $(-2c + 2d)/\sqrt{2}$.

(iii) $-8/\sqrt{14}$.

(iv) $(1 + \pi)/12$.

(v) $2/\sqrt{21} + 13/\sqrt{210}$.

解析学 III 演習 No. ４

［１］ $z = f(x, y)$ が２階連続微分可能で，$x = a + ut, y = b + vt$ であるとき，

$$\frac{dz}{dt} = uf_x + vf_y, \quad \frac{d^2 z}{dt^2} = u^2 f_{xx} + 2uvf_{xy} + v^2 f_{yy}$$

が成り立つことを証明せよ．ただし，$a, b, u, v \in \mathbb{R}$ は定数.

［２］ $z = f(x, y)$ が２階連続微分可能で，$x = r\cos\theta, y = r\sin\theta$ であるとき，次を示せ.

(i) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = r\dfrac{\partial z}{\partial r}.$

(ii) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = \left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2.$

(iii) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r}\left(\dfrac{\partial z}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 z}{\partial \theta^2}\right).$

解析学 III 演習 No. ４ の解答例

［１］ (i)

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=uf_x+vf_y.$$

(ii)

$$\begin{aligned}\frac{d^2z}{dt^2} &= \frac{\partial(uf_x+vf_y)}{\partial x}\frac{dx}{dt}+\frac{\partial(uf_x+vf_y)}{\partial y}\frac{dy}{dt}\\ &= (uf_{xx}+vf_{yx})u+(uf_{xy}+vf_{yy})v\\ &= u^2f_{xx}+2uvf_{xy}+v^2f_{yy}.\end{aligned}$$

［２］ (i)

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y}.$$

故に，

$$r\frac{\partial z}{\partial r}=r\cos\theta\frac{\partial z}{\partial x}+r\sin\theta\frac{\partial z}{\partial y}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}.$$

(ii)

$$\left(\frac{\partial z}{\partial r}\right)^2=\cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2+\sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2+2\cos\theta\sin\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y},$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=(-r\sin\theta)\frac{\partial z}{\partial x}+(r\cos\theta)\frac{\partial z}{\partial y}.$$

故に，

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2+\cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2-2\cos\theta\sin\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}.$$

よって，

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$

(iii)

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \cos\theta\frac{\partial}{\partial x}\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\sin\theta\frac{\partial}{\partial x}\frac{\partial z}{\partial y}\frac{\partial x}{\partial r}\\ &\quad+\cos\theta\frac{\partial}{\partial y}\frac{\partial z}{\partial x}\frac{\partial y}{\partial r}+\sin\theta\frac{\partial}{\partial y}\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\\ &= \cos^2\theta\frac{\partial^2 z}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 z}{\partial y^2}.\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 z}{\partial \theta^2} &= (-r\cos\theta)\frac{\partial z}{\partial x} + (-r\sin\theta)\frac{\partial}{\partial \theta}\frac{\partial z}{\partial x} + (-r\sin\theta)\frac{\partial z}{\partial y} + (r\cos\theta)\frac{\partial}{\partial \theta}\frac{\partial z}{\partial y} \\
&= (-r\cos\theta)\frac{\partial z}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial \theta}\right) \\
&\quad +(-r\sin\theta)\frac{\partial z}{\partial y} + (r\cos\theta)\left(\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial \theta^2}\right) \\
&= (r^2\sin^2\theta)\frac{\partial^2 z}{\partial x^2} + (-2r^2\sin\theta\cos\theta)\frac{\partial^2 z}{\partial x \partial y} \\
&\quad +(r^2\cos^2\theta)\frac{\partial^2 z}{\partial y^2} - r\left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right).
\end{aligned}
$$

故に,

$$
\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\left(\frac{\partial z}{\partial r} + \frac{1}{r}\frac{\partial^2 z}{\partial \theta^2}\right) = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.
$$

解析学 III 演習 No. ５

［１］ 次の２変数関数について，

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y)$$

を求めよ．

(i) $f(x,y)=\sin x\sin y$.

(ii) $f(x,y)=(x-y)/(x+y)$.

［２］ 次の３変数関数について，

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}+l\frac{\partial}{\partial z}\right)^2 f(x,y,z)$$

を求めよ．

(i) $f(x,y,z)=x^3+y^3+z^3$.

(ii) $f(x,y,z)=xze^{x^2-y^2}$.

(iii) $f(x,y,z)=\cos(x-yz)$.

［３］ 次の２変数関数を点 $(0,0)$ について，２次のテイラー多項式 P_2 とその剰余項 R を求めよ．

(i) $f(x,y)=\cos(x+y)$.

(ii) $f(x,y)=\cos x\sin y$.

(iii) $f(x,y)=ye^{x-y}$.

(iv) $f(x,y)=\sin(x+y)$.

(v) $f(x,y)=\sin x+\cos y$.

解析学 III 演習 No. ５ の解答例

［１］ (i) $h^2(-\sin x\sin y)+2hk\cos x\cos y+k^2(-\sin x\sin y)$.
(ii) $h^2\frac{(-4y)}{(x+y)^3}+2hk\frac{2(x-y)}{(x+y)^3}+k^2\frac{4x}{(x+y)^3}$.

［２］ (i) $6(h^2x+k^2y+l^2z)$.
(ii)

$$2xz(3+2x^2)e^{x^2-y^2}h^2-4yz(1+2x^2)e^{x^2-y^2}hk+2xz(2y^2-1)e^{x^2-y^2}k^2 \\ +2(1+2x^2)e^{x^2-y^2}hl-4xye^{x^2-y^2}kl.$$

(iii)

$$-\cos(x-yz)h^2+2z\cos(x-yz)hk-z^2\cos(x-yz)k^2 \\ +2y\cos(x-yz)hl-y^2\cos(x-yz)l^2+2(\sin(x-yz)-yz\cos(x-yz)kl.$$

［３］ (i) $P_2=1-\frac{1}{2}(x^2+2xy+y^2)$. $R=\frac{1}{6}(x^2+3x^2y+3xy^2+y^3)\sin(\theta x+\theta y)$.
(ii) $P_2=y$. $R=\frac{1}{6}(x^3\sin\theta x\sin\theta y-3x^2y\cos\theta x\cos\theta y+3xy^2\sin\theta x\sin\theta y-y^3\cos\theta x\cos\theta y)$.
(iii) $P_2=y+xy-y^2$. $R=\frac{1}{6}\{x^3\theta y+3x^2y(1-\theta y)+3xy^2(-2+\theta y)+y^3(3-\theta y)\}e^{\theta x-\theta y}$.
(iv) $P_2=x+y$. $R=-\frac{1}{6}(x^3+3x^2y+3xy^2+y^3)\cos(\theta x+\theta y)$.
(v) $P_2=1+x-\frac{y^2}{2}$. $R=\frac{1}{6}(-x^3\cos\theta x+y^3\sin\theta y)$

解析学 III 演習 No. ６

［１］ 次の $\mathbb{R}^2$ で定義された２変数関数について，極大点，極小点，鞍点を求めよ．また，最大値，最小値が存在する場合はそれらを求めよ．

(i) $f(x,y) = 5 + x + y - x^2 - y^2$.

(ii) $f(x,y) = x^2 + xy$.

(iii) $f(x,y) = x^2 - xy + y^2 + 3x - y + 4$.

(iv) $f(x,y) = 3xy - x^4 - y^4 + 2$.

(v) $f(x,y) = \dfrac{1}{x^2 + y^2 + 1}$.

(vi) $f(x,y) = \dfrac{1}{x} + 2xy + \dfrac{1}{y}$.

(vii) $f(x,y) = y \sin x$.

(viii) $f(x,y) = e^{-x^2+y^2-2y}$.

(ix) $f(x,y) = xy \log(x^2 + y^2)$.

解析学 III 演習 No. ６ の解答例

［１］ (i) $f_x = 1-2x = 0, f_y = 1-2y = 0$ を解いて，$x = y = 1/2$. $f_{xx} = -2, f_{xy} = 0, f_{yy} = -2$.

$$H_f(1/2, 1/2) = f_{xx}(1/2, 1/2)f_{yy}(1/2, 1/2) - f_{xy}(1/2, 1/2)^2 = 4 > 0.$$

$f_{xx}(1/2, 1/2) = -2 < 0$ だから，$(x, y) = (1/2, 1/2)$ で極大．唯一の極大点であるから，そこで最大となり，最大値は 11/2.
(ii) $f_x = 2x + y = 0, f_y = x = 0$ を解いて，$x = y = 0$. $f_{xx} = 2, f_{xy} = 1, f_{yy} = 0$.

$$H_f(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = -1 < 0$$

だから，$(x, y) = (0, 0)$ は鞍点．
(iii) $f_x = 2x-y+3 = 0, f_y = -x+2y-1 = 0$ より，$x = -5/3, y = -1/3$. $f_{xx} = 2, f_{xy} = -1, f_{yy} = 2$. $H_f(-5/3, -1/3) = 3 > 0$. $f_{xx}(-5/3, -1/3) = 2 > 0$ だから，$(x, y) = (-5/3, -1/3)$ で極小．唯一の極小だから，最小でもあり，最小値は $f(-5/3, -1/3) = 5/3$.
(iv) $f_x = 3y - 4x^3 = 0, f_y = 3x - 4y^3 = 0$ より，$(x, y) = (0, 0)$, $(\sqrt{3}/2, \sqrt{3}/3)$, $(-\sqrt{3}/2, -\sqrt{3}/2)$. $f_{xx} = -12x^2, f_{xy} = 3, f_{yy} = -12y^2$.

$H_f(0, 0) = -9 < 0$ だから，$(x, y) = (0, 0)$ は鞍点．

$H_f(\sqrt{3}/2, \sqrt{3}/2) = 72 > 0$. $f_{xx}(\sqrt{3}/2, \sqrt{3}/2) = -9 < 0$ だから，$(\sqrt{3}/2, \sqrt{3}/2)$ で極大．

$H_f(-\sqrt{3}/2, -\sqrt{3}/2) = 72 > 0$. $f_{xx}(-\sqrt{3}/2, -\sqrt{3}/2) = -9 < 0$ だから，$(-\sqrt{3}/2, -\sqrt{3}/2)$ で極大．

(v) $f_x = (-1)(x^2 + y^2 + 1)^{-2}2x = 0, f_y = (-1)(x^2 + y^2 + 1)^{-2}2y = 0$ より，$(x, y) = (0, 0)$. $f_{xx} = 2(x^2 + y^2 + 1)^{-3}4x^2 - 2(x^2 + y^2 + 1)^{-2}, f_{yy} = 2(x^2 + y^2 + 1)^{-3}4y^2 - 2(x^2 + y^2 + 1)^{-2}, f_{xy} = 2(x^2 + y^2 + 1)^{-3}4xy$.

$H_f(0, 0) = 4 > 0, f_{xx}(0, 0) = -2 < 0$ だから，$(x, y) = (0, 0)$ で極大．唯一の極大点であるから最大で，最大値は $f(0, 0) = 1$.

(vi) $f_x = \frac{-1}{x^2} + 2y = 0, f_y = 2x + \frac{-1}{y^2} = 0$ より，$2x^2y = 1, 2xy^2 = 1$. $x, y > 0$ より $x = y = 1/\sqrt[3]{2}$. $f_{xx} = 2/x^3, f_{yy} = 2/y^3, f_{xy} = 2$. $H_f(1/\sqrt[3]{2}, 1/\sqrt[3]{2}) = 12 > 0$. $f_{xx}(1/\sqrt[3]{2}, 1/\sqrt[3]{2}) = 4 > 0$ だから，$(x, y) = (1/\sqrt[3]{2}, 1/\sqrt[3]{2})$ で極小．唯一の極小点であるから，最小．最小値は，$3\sqrt[3]{2}$.

(viii) $f_x = -2xe^{-x^2+y^2-2y} = 0, f_y = (2y-2)e^{-x^2+y^2-2y} = 0$ より，$x = 0, y = 1$. $f_{xx} = -2e^{-x^2+y^2-2y} + 4x^2e^{-x^2+y^2-2y}$, $f_{yy} = 2e^{-x^2+y^2-2y} + (2y-2)^2e^{-x^2+y^2-2y}$, $f_{xy} = -2x(2y-2)e^{-x^2+y^2-2y}$.

$H_f(0,1) = -4e^{-2} < 0$. よって，$(0,1)$ は鞍点.

(ix) $f_x = y\log(x^2+y^2) + \frac{2x^2y}{x^2+y^2} = 0, f_y = x\log(x^2+y^2) + \frac{2xy^2}{x^2+y^2} = 0$ より，

$$(x,y) = (\pm 1, 0), (0, \pm 1), (1/\sqrt{2e}, 1/\sqrt{2e}), (1/\sqrt{2e}, -1/\sqrt{2e}),\\ (-1/\sqrt{2e}, 1/\sqrt{2e}), (-1/\sqrt{2e}, -/\sqrt{2e}).$$

$f_{xx} = \frac{2x^3y+6xy^3}{(x^2+y^2)^2}$, $f_{yy} = \frac{6x^3y+2xy^3}{(x^2+y^2)^2}$, $f_{xy} = \log(x^2+y^2) + \frac{2(x^4+y^4}{(x^2+y^2)^2}$.

$H_f(\pm 1, 0) = -4 < 0$, $H_f(0, \pm 1) = -4 < 0$. $(\pm 1, 0), (0, \pm 1)$ では鞍点.

$H_f(\pm 1/\sqrt{2e}, \pm 1/\sqrt{2e}) = 4 > 0$. $f_{xx}(\pm 1/\sqrt{2e}, \pm 1/\sqrt{2e}) = 2 > 0$ （復号同順）. よって，$(\pm 1/\sqrt{2e}, \pm 1/\sqrt{2e})$ で極小.

$H_f(\pm 1/\sqrt{2e}, \mp 1/\sqrt{2e}) = 4 > 0$. $f_{xx}(\pm 1/\sqrt{2e}, \mp 1/\sqrt{2e}) = -2 < 0$ （復号同順）. よって，$(\pm 1/\sqrt{2e}, \mp 1/\sqrt{2e})$ で極大.

解析学 III 演習 No. 7

［1］ y が x の関数であり，次が成り立つとき，y の極値を求めよ．

(i) $xy^2 - x^2y = 16.$

(ii) $x^3 + y^3 - x - y = 0.$

(iii) $x^3 - 3axy + y^3 = 0 \quad (a > 0).$

解析学 III 演習 No. 7 解答例

［１］ (i) $F = xy^2 - x^2y - 16 = 0, F_x = y^2 - 2xy = 0$ をみたす点は $(x, y) = (2, 4)$ である．このとき，陰関数を $y = f(x)$ とし，

$$f''(2) = -\frac{F_{xx}(2,4)}{F_y(2,4)} = \frac{2}{3} > 0$$

よって $y = f(x)$ は $x = 2$ で極小値 4 をとる．

(ii) $F = x^3+y^3-x-y = (x+y)(x^2-xy+y^2-1) = 0, F_x = 3x^2-1 = 0$ をみたす点は $(x, y) = (1/\sqrt{3}, -1/\sqrt{3}), (1/\sqrt{3}, 2/\sqrt{3}), (-1/\sqrt{3}, 1/\sqrt{3}), (-1/\sqrt{3}, -2/\sqrt{3})$ である．このとき，$(1/\sqrt{3}, -\sqrt{3}), (-1/\sqrt{3}, 1/\sqrt{3})$ では，$F_y = 0$ となるので除外する．

$$f''(1/\sqrt{3}) = -\frac{F_{xx}(1/\sqrt{3}, 2/\sqrt{3})}{F_y(1/\sqrt{3}, 2/\sqrt{3})} = -2/\sqrt{3} < 0$$

よって y は $x = 1/\sqrt{3}$ で極大値 $2/\sqrt{3}$ をとる．

$$f''(-\sqrt{3}) = -\frac{F_{xx}(-1/\sqrt{3}, -2/\sqrt{3})}{F_y(-1/\sqrt{3}, -2/\sqrt{3})} = 2/\sqrt{3} > 0$$

よって y は $x = -1/\sqrt{3}$ で極小値 $-2/\sqrt{3}$ をとる．

(iii) $F = x^3 - 3axy + y^3 = 0, F_x = 3x^2 - 3ay = 0$ より，$(x, y) = (0,0), (2^{1/3}a, 2^{2/3}a)$. 原点では，$F_y = 0$ となる．

$$-\frac{F_{xx}(2^{1/3}, 2^{2/3})}{F_y(2^{1/3}, 2^{2/3})} = \frac{-2}{a} < 0.$$

よって，$x = 2^{1/3}$ で極大値 $2^{2/3}$ をとる．

解析学 III 演習 No. ８

［１］次の重積分を累次積分により求めよ．ただし，$\displaystyle\iint_{[a,b]\times[c,d]} f(x,y)dxdy$ を $\displaystyle\int_c^d \int_a^b f(x,y)dxdy$ と書く．

(i) $\displaystyle\int_1^2 \int_0^1 (1+xy)dxdy$

(ii) $\displaystyle\int_0^1 \int_1^2 (1+xy)dydx$

(iii) $\displaystyle\int_1^2 \int_0^1 dxdy$

(iv) $\displaystyle\int_4^7 \int_{-3}^1 dydx$

(v) $\displaystyle\int_1^2 \int_{-1}^1 e^{u+v}dudv$

(vi) $\displaystyle\int_{-1}^1 \int_1^2 e^{u+v}dudv$

(vii) $\displaystyle\int_1^3 \int_1^2 2y\log x dydx$

(viii) $\displaystyle\int_0^{\pi/2} \int_0^{\pi/2} (\cos(x+t)dxdt$

(ix) $\displaystyle\int_\pi^{2\pi} \int_0^\pi (\sin x+\cos y)dxdy$

(x) $\displaystyle\int_0^2 \int_0^\pi \cos x dydx$

解析学 III 演習 No. ８ 解答例

［１］ (i) $7/4$.

(ii) $7/4$.

(iii) 1.

(iv) 12.

(v) $e^3 - e^2 - e + 1$.

(vi) $e^3 - e^2 - e + 1$.

(vii) $9 \log 3 - 6$.

(viii) 0.

(ix) 2π.

(x) $\pi \sin 2$.

解析学 III 演習 No. ９

［１］ 次の重積分を求めよ.

(i) $\displaystyle\int_0^2 \int_x^{x^2} dydx.$

(ii) $\displaystyle\int_0^2 \int_0^{x} xydydx.$

(iii) $\displaystyle\int_0^\pi \int_0^{x} x\sin ydydx.$

(iv) $\displaystyle\int_0^1 \int_x^{x^2} (2x-y)dydx.$

(v) $\displaystyle\int_1^2 \int_0^{1/y} xe^{xy}dxdy.$

(vi) $\displaystyle\int_0^1 \int_{y^2}^{y} \sqrt{xy}dxdy.$

(vii) $\displaystyle\int_0^1 \int_0^{y} x\sqrt{y^2-x^2}dxdy.$

(viii) $\displaystyle\int_1^e \int_1^{\log y} e^x dxdy.$

(ix) $\displaystyle\int_0^\pi \int_0^{\cos y} x\sin ydxdy.$

解析学 III 演習 No. ９ 解答例

［１］ (i) $2/3$.
(ii) 2.
(iii) $\pi^2/2+2$.
(iv) $-1/10$.
(v) $1/2$.
(vi) $2/27$.
(vii) $1/12$.
(viii) $-e^2/2+e-1/2$.
(ix) $1/3$.

解析学 III 演習 No. １０

［１］ $f(x,y)=(x+y)^p$ の正方形 $G=\{(x,y); a\leq x\leq b, a\leq y\leq b\}$ での重積分を求めよ．ただし，$a>0, p\neq -2, p\neq -1$ とする．

［２］G を直線 $x+y=2$ と放物線 $y=x^2$ とで囲まれる領域とする．このとき，

$$I=\iint_G \sqrt{y-x^2}dxdy$$

を求めよ．ただし，$\int_0^{\pi/2}\cos^4\theta d\theta=\frac{3}{16}\pi$ は使ってよい．

［３］$p>-2, p\neq -1$, $G=\{(x,y); 0<x\leq 1, 0<y\leq 1\}$ のとき，広義積分

$$\iint_G (x+y)^p dxdy$$

を求めよ．

解析学 III 演習 No. １０ 解答例

［１］

$$\begin{aligned}\iint_G (x+y)^p dxdy &= \int_a^b dx \int_a^b (x+y)^p dy \\ &= \int_a^b \left[\frac{(x+y)^{p+1}}{p+1}\right]_{y=a}^{y=b} dx \\ &= \frac{1}{p+1}\int_a^b \{(x+b)^{p+1}-(x+a)^{p+1}\}dx \\ &= \frac{1}{(p+1)(p+2)}\{(2b)^{p+2}+(2a)^{p+2}-2(a+b)^{p+2}\}.\end{aligned}$$

［２］

$$\begin{aligned} I &= \int_{-2}^1 dx \int_{x^2}^{2-x} \sqrt{y-x^2}dy \\ &= \frac{2}{3}\int_{-2}^1 (2-x-x^2)^{3/2}dx \\ &= \frac{2}{3}\int_{-2}^1 \left((\frac{3}{2})^2-(x+\frac{1}{2})^2\right)^{3/2} dx \\ &= \frac{2}{3}\int_{-3/2}^{3/2}\left((\frac{3}{2})^2-t^2\right)^{3/2} dt \qquad (t=x+\frac{1}{2}) \\ &= \frac{2}{3}\left(\frac{3}{2}\right)^4\int_{-1}^1 (1-s^2)^{3/2}ds \\ &= \frac{2}{3}\left(\frac{3}{2}\right)^4 2\int_0^1 (1-s^2)^{3/2}ds \qquad (\text{被積分関数は偶関数}) \\ &= \frac{4}{3}\left(\frac{3}{2}\right)^4\int_0^{\pi/2} \cos^4\theta d\theta \qquad (s=\sin\theta) \\ &= \frac{81}{64}\pi.\end{aligned}$$

［３］$G_n=\{(x,y);1/n\le x\le 1, 1/n\le y\le 1\}$ は G に対する一つの近似増加列である. ［１］より

$$\iint_{G_n}(x+y)^p dxdy = \frac{1}{(p+1)(p+2)}\left\{2^{p+2}+(\frac{2}{n})^{p+2}-2(\frac{n+1}{n})^{p+2}\right\}.$$

$n\to\infty$ として，$\iint_G (x+y)^p dxdy = \frac{2^{p+2}-2}{(p+1)(p+2)}$.